

# Boundary Singularities on a Wedge-like Domain of a Semilinear Elliptic Equation

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## Abstract

Let  $n \geq 2$  and  $\Omega \subset \mathbb{R}^{n+1}$  be a Lipschitz wedge-like domain. We construct positive weak solutions of the problem

$$\Delta u + u^p = 0 \quad \text{in } \Omega,$$

which vanish in a suitable trace sense on  $\partial\Omega$ , but which are singular at prescribed “edge” of  $\Omega$  if  $p$  is equal or slightly above a certain exponent  $p_0 > 1$  which depends on  $\Omega$ . Moreover, in the case which  $\Omega$  is unbounded, the solutions have fast decay at infinity.

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## 1 Introduction

Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$ ,  $n \geq 2$  with smooth boundary  $\partial\Omega$ . A model of nonlinear elliptic boundary value problem is the classical Lane-Emden-Fowler equation,

$$\begin{cases} -\Delta u &= |u|^p & \text{in } & \Omega, \\ u &> 0 & \text{in } & \Omega, \\ u &= 0 & \text{in } & \partial\Omega, \end{cases} \quad (1.1)$$

where  $p > 1$ . Following Brezis and Turner [3] and Quittner and Souplet [13], we will say that a positive function  $u$  is a very weak solution of problem (1.1), if  $u$  and  $\text{dist}(x, \partial\Omega)u^p \in L^1(\Omega)$ , and

$$\int_{\Omega} u \Delta v + |u|^p v dx = 0, \quad \forall v \in C^2(\overline{\Omega}), \text{ with } v = 0 \text{ on } \partial\Omega.$$

From the results in [3, 13], it follows that if  $p$  satisfies the constraint

$$1 < p < \frac{n+1}{n-1}, \quad (1.2)$$

then  $u \in C^2(\overline{\Omega})$ , i.e.  $u$  is a classical solution of problem (1.1).

It is well known that, if  $1 < p < \frac{n+2}{n-2}$ , one can use Sobolev’s embedding and standard variational techniques to prove the existence of a positive very weak solution of problem (1.1). However, if  $\frac{n+1}{n-1} < p < \frac{n+2}{n-2}$ , this very weak solution may not be bounded. A result in the understanding of very weak solutions was achieved by Souplet [14]. He constructed an example of a positive function  $a \in L^\infty(\Omega)$  such that problem (1.1), with  $u^p$  replaced by  $a(x)u^p$  for  $p > \frac{n+1}{n-1}$ , has a very weak

solution which is unbounded, developing a point singularity on the boundary. This shows that the exponent  $p = \frac{n+1}{n-1}$  is truly a critical exponent. Let us mention that the study of the behavior near an isolated boundary singularity of any positive solution of (1.1) when the exponent  $p \geq \frac{n+1}{n-1}$  was achieved by Bidaut-Véron-Ponce-Véron in [2]. Finally, del Pino-Musso-Pacard [5] showed the existence of  $\varepsilon > 0$  such that for any  $p \in [\frac{n+1}{n-1}, \frac{n+1}{n-1} + \varepsilon)$  an unbounded, positive, very weak solution of (1.1) exists which blows up at a prescribed point of  $\partial\Omega$ . For the respective problem with interior singularity see for example [4, 6, 11, 12].

Let us give some definitions for convenience to the reader. Let  $n \geq 2$  and  $(r, \theta) \in [0, \infty) \times \mathbb{S}^{n-1}$  be the spherical-coordinates of  $x \in \mathbb{R}^n$  abbreviated by  $x = (r, \theta)$ . Given an open Lipschitz spherical cap  $\omega \subsetneq \mathbb{S}^{n-1}$  let

$$C_\omega = \{x = (r, \theta) : r > 0, \theta \in \omega\},$$

be the corresponding infinite cone. The set

$$C_\omega^R = C_\omega \cap B_R(0) \subset \mathbb{R}^n$$

is called a conical piece with spherical cap  $\omega$  and radius  $R$ .

A bounded Lipschitz domain  $\Omega \subset C_\omega$  is called a domain with a conical boundary piece if there exists a conical piece  $C_\omega^R$  such that  $\Omega \cap B_R(0) = C_\omega^R$ .

We denote by  $\lambda$  and  $\phi_1(\theta)$  to be respectively the first eigenvalue and the corresponding eigenfunction of the problem

$$\begin{cases} -\Delta_{\mathbb{S}^{n-1}} u &= \lambda u & \text{in } \omega \\ u &= 0 & \text{on } \partial\omega, \end{cases} \quad (1.3)$$

with  $\int_\omega \phi_1^2 dS_x = 1$ .

Finally, we define the exponent

$$p^* = \frac{n + \gamma}{n + \gamma - 2}; \quad \text{with} \quad \gamma = \frac{2 - n}{2} + \sqrt{\left(\frac{n - 2}{2}\right)^2 + \lambda}, \quad (1.4)$$

and note that  $p^*$  depends on  $\omega$ .

In the same spirit as above, McKennab-W. Reichel [9] generalized the results of Souplet [14] to domain with conical boundary piece, and they showed that the exponent  $p^*$  is a truly critical exponent, in the sense that, if  $1 < p < p^*$ , then every very weak solution of problem (1.1) is bounded (see also [1]). Finally, Horák-McKennab-Reichel [8] considered a bounded Lipschitz domain  $\Omega$  with a conical boundary piece of spherical cap  $\omega \subset \mathbb{S}^{n-1}$ , at  $0 \in \partial\Omega$ , and they proved the existence of  $\varepsilon > 0$  such that for any  $p \in (p^*, p^* + \varepsilon)$  an unbounded, positive, very weak solution of (1.1) exists which blows up at  $0 \in \partial\Omega$ .

Let us consider the following problem

$$\begin{cases} \Delta_x u + u^p &= 0, & \text{in } C_\omega \\ u &> 0, & \text{in } C_\omega \\ u &= 0, & \text{on } \partial C_\omega \setminus \{0\}. \end{cases} \quad (1.5)$$

The authors in [8] proved that problem (1.5) admits a positive solution of the form  $w(\theta) = |x|^{-\frac{2}{p-1}} \phi_p(\theta)$ , where  $\phi_p$  solves the problem

$$\begin{aligned} \Delta_{\mathbb{S}^{n-1}} \phi - \frac{2}{p-1} \left( -\frac{2}{p-1} + n - 2 \right) \phi + \phi^p &= 0, & \text{in } \omega \\ \phi &= 0, & \text{on } \partial\omega, \end{aligned} \quad (1.6)$$

for any  $p \in (p^*, \infty)$  if  $n = 2, 3$  and any  $p \in (p^*, \frac{n+1}{n-3})$  if  $n \geq 4$ . But this solution does not have fast decay at infinity.

We note here that if  $\omega = \mathbb{S}_+^{n-1}$ , then  $\gamma = 1$ , thus the critical exponent  $p^* = \frac{n+1}{n-1}$  and  $C_\omega = \mathbb{R}_+^n$ . In [5], del Pino-Musso-Pacard constructed a solution of problem (1.5) in  $\mathbb{R}_+^n$  with fast decay. More precisely they showed that there exists  $\varepsilon > 0$  such that for any  $p \in (\frac{n+1}{n-1}, \frac{n+1}{n-1} + \varepsilon)$  problem (1.5) in  $\mathbb{R}_+^n$  admits a solution  $u \in C^2(\mathbb{R}_+^n)$  satisfying

$$u(x) \approx |x|^{-\frac{2}{p-1}} \phi_p(\theta), \quad \text{as } |x| \rightarrow 0$$

and

$$u(x) \approx |x|^{-(n-1)} \phi_1(\theta), \quad \text{as } |x| \rightarrow \infty.$$

The first result of this work is the construction of a singular solution at 0 with fast decay at infinity, for problem (1.5). In particular we prove

**Theorem 1.1.** *There exists a number  $p(n, \lambda) > p^*$ , such that for any*

$$p \in (p^*, p(n, \lambda)),$$

*there exists a solution  $u_1(x)$  to problem (1.5) such that*

$$u_1(x) = |x|^{-\frac{2}{p-1}} \phi_p(\theta)(1 + o(1)) \quad \text{as } |x| \rightarrow 0,$$

*where  $\phi_p$  solves (1.6), and*

$$u_1(x) = |x|^{2-\gamma-n} \phi_1(\theta)(1 + o(1)) \quad \text{as } |x| \rightarrow \infty,$$

*where  $\gamma$  is defined in (1.4). In addition, we have the pointwise estimate*

$$|u_1(x)| \leq C|x|^{-\frac{2}{p-1}} \|\phi_p\|_{C^2(\omega)},$$

*for some constant  $C > 0$  which does not depend on  $p$ .*

To describe our main result let us introduce some new notations.

Let  $x \in \mathbb{R}^n$  with  $n \geq 2$ . Given  $\tau \in \mathbb{R}$ , we let  $\omega(\tau) \subsetneq \mathbb{S}^{n-1}$  to be the corresponding Lipschitz spherical cap. We set

$$r_{\sigma(\tau)} = |x - \sigma(\tau)|,$$

where  $\sigma : \mathbb{R} \rightarrow \mathbb{R}^n$  is a smooth curve such that

$$\sup_{\tau \in \mathbb{R}} \{|\sigma(\tau)| + |\sigma'(\tau)| + |\sigma''(\tau)|\} < C < \infty.$$

Now, given  $\tau$ , we let  $(r_{\sigma(\tau)}, \theta) \in [0, \infty) \times \mathbb{S}^{n-1}$  to be the spherical-coordinates of  $x \in \mathbb{R}^n$  centered at  $\sigma(\tau)$  abbreviated by  $x = (r_{\sigma(\tau)}, \theta)$ . We define

$$\tilde{C}_{\omega(\tau)} = \{x = (r_{\sigma(\tau)}, \theta) : r_{\sigma(\tau)} > 0, \theta \in \omega(\tau)\} \subset \mathbb{R}^n$$

and we set

$$\Omega_{\tau_1, \tau_2} = \{(\tau, x) \in (\tau_1, \tau_2) \times \mathbb{R}^n : x \in \tilde{C}_{\omega(\tau)}\} \subset \mathbb{R}^{n+1},$$

$$\Omega_{\tau_1, \tau_2}^R = \Omega_{\tau_1, \tau_2} \cap \{(\tau, x) \in (\tau_1, \tau_2) \times \mathbb{R}^n : x \in B_R(\sigma(\tau))\} \subset \mathbb{R}^{n+1},$$

and

$$S_{\tau_1, \tau_2} = \{(\tau, x) \in [\tau_1, \tau_2] \times \mathbb{R}^n : r_{\sigma(\tau)} = 0\}.$$

Finally we define  $\lambda^* = \inf_{\tau \in \mathbb{R}} \lambda(\tau)$  and  $\gamma^* = \inf_{\tau \in \mathbb{R}} \gamma(\tau)$ .

In this work we assume that  $\omega(\tau)$  depends smoothly on  $\tau$ , i.e.  $\lambda(\tau)$  is a smooth bounded function with respect  $\tau$  with bounded derivatives. We also assume that  $\inf_{\tau \in \mathbb{R}} \lambda(\tau) > 0$ . Finally, we suppose that there exists  $\varepsilon > 0$ , such that for any  $p \in \left( \sup_{\tau \in \mathbb{R}} p^*(\tau), \sup_{\tau \in \mathbb{R}} p^*(\tau) + \varepsilon \right)$ , there exists a solution  $u_1(\tau, x)$  of theorem 1.1. That means,  $\text{osc}_{\tau \in \mathbb{R}} \lambda(\tau)$  is small enough.

**Theorem 1.2.** *Let  $\varepsilon > 0$  be small enough. Then there exists a number  $p_0 > \sup_{\tau \in \mathbb{R}} p^*$  such that, given  $p \in (\sup_{\tau \in \mathbb{R}} p^*, p_0)$ , and  $\frac{2}{p-1} \leq -\rho < n + \gamma^* - 2$ , the following problem*

$$\begin{cases} -\Delta u &= u^p & \text{in} & \Omega_{-\infty, \infty}, \\ u &> 0 & \text{in} & \Omega_{-\infty, \infty} \\ u &= 0 & \text{on} & \partial\Omega_{-\infty, \infty} \setminus S_{-\infty, \infty} \end{cases}$$

*possesses very weak solutions  $u$ . In addition we have that*

$$u(\tau, x) \approx u_1 \left( \tau, \frac{x - \sigma(\tau)}{\varepsilon} \right) \quad \text{as } r_{\sigma(\tau)} \rightarrow 0,$$

*where  $u_1$  is in theorem 1.1. And*

$$u(\tau, x) \leq C r_{\sigma(\tau)}^\rho \quad \text{as } r_{\sigma(\tau)} \rightarrow \infty.$$

Our third and final result of this paper is the following

**Theorem 1.3.** *Let  $\alpha > 0$  be small enough and  $\Omega \subset \mathbb{R}^{n+1}$  be a bounded Lipschitz domain such that*

$$\Omega \cap \Omega_{\tau_1 - \alpha, \tau_2 + \alpha}^R = \Omega_{\tau_1 - \alpha, \tau_2 + \alpha}^R \subset \mathbb{R}^{n+1}.$$

*There exists a number  $p_0 > \sup_{\tau \in \mathbb{R}} p^*$  such that, given  $p \in (\sup_{\tau \in \mathbb{R}} p^*, p_0)$ , there exist very weak solutions  $u$  to the problem*

$$\begin{cases} -\Delta u &= u^p, & \text{in} & \Omega, \\ u &> 0, & \text{in} & \Omega \\ u &= 0, & \text{on} & \partial\Omega \setminus S_{\tau_1 - \alpha, \tau_2 + \alpha}. \end{cases}$$

*Moreover,  $\forall (\tau, x) \in \Omega_{\tau_1 - \frac{\alpha}{4}, \tau_2 + \frac{\alpha}{4}}^R$*

$$u(\tau, x) \approx u_1 \left( \tau, \frac{x - \sigma(\tau)}{\varepsilon} \right) \quad \text{as } r_{\sigma(\tau)} \rightarrow 0.$$

The paper is organized as follows. In section 3 we prove theorem 1.1. In subsection 3.1, we prove some regularity results with respect  $\tau$ , for the function  $u_1(\tau, x)$  in theorem 1.1. Section 4 will be devoted to the proofs of theorems 1.2 and 1.3.

## 2 The eigenvalue problem on spherical caps.

Let  $n \geq 2$ ,  $\tau \in \mathbb{R}$ , and  $\omega(\tau) \subsetneq \mathbb{S}^{n-1}$  be the corresponding open Lipschitz spherical cap. We denote by  $\lambda(\tau)$  and  $\phi_1(\tau, \theta)$  to be respectively the first eigenvalue and eigenfunction of the eigenvalue problem

$$\begin{cases} -\Delta_{\mathbb{S}^{n-1}} u = \lambda(\tau)u, & \text{in } \omega(\tau) \\ u = 0, & \text{on } \partial\omega, \end{cases} \quad (2.1)$$

with  $\int_{\omega(\tau)} \phi_1^2 dS_x = 1$ .

We assume that  $\omega(\tau)$  depends smoothly on  $\tau$ , i.e.  $\lambda(\tau)$  is a smooth bounded function with respect  $\tau$  with bounded derivatives. We also assume that  $\inf_{\tau \in \mathbb{R}} \lambda(\tau) > 0$ .

Now note that, without loss of generality, we can set  $\theta_1 = \cos t$ , with  $0 < t < \beta(\tau)$ , where  $\beta(\tau)$  is a smooth function with bounded derivatives satisfying

$$\begin{cases} 0 < \inf_{\tau \in \mathbb{R}} \beta(\tau) < \sup_{\tau \in \mathbb{R}} \beta(\tau) < 2\pi & \text{for } n = 2 \\ \text{and} \\ 0 < \inf_{\tau \in \mathbb{R}} \beta(\tau) < \sup_{\tau \in \mathbb{R}} \beta(\tau) < \pi & \text{for } n \geq 3. \end{cases}$$

Then problem (2.1) is equivalent to the following one

$$\begin{cases} -\sin^{2-n} t \frac{d}{dt} \left( \sin^{n-2} t \frac{d\phi_1}{dt} \right) = \lambda \phi_1 & \text{in } (0, \beta(\tau)). \\ \phi_1(\beta(\tau)) = 0 \\ \frac{d\phi_1}{dt}(0) = 0, \end{cases} \quad (2.2)$$

with

$$C(n) \int_0^{\beta(\tau)} \sin^{n-2}(t) |u|^2 dt = \int_{\omega} |\phi_1|^2 dS = 1.$$

We note here that, for  $n = 2$  in problem (2.2), we may have  $\phi_1(0) = 0$  instead of  $\frac{d\phi_1}{dt}(0) = 0$ .

We have the following lemma

**Lemma 2.1.** *Let  $\phi_1(\tau, \theta)$  be the first eigenfunction of the following eigenvalue problem*

$$\begin{cases} -\Delta_{\mathbb{S}^{n-1}} u = \lambda u, & \text{in } \omega(\tau) \\ u = 0, & \text{on } \partial\omega(\tau), \end{cases} \quad (2.3)$$

*with  $\int_{\omega(\tau)} \phi_1^2 dS = 1$ . Then there exists a positive constant  $C$  such that*

$$\sup_{\tau \in \mathbb{R}} \left\| |\phi_1| + \left| \frac{\partial \phi_1}{\partial \tau} \right| + \left| \frac{\partial^2 \phi_1}{\partial \tau^2} \right| \right\|_{L^\infty(\omega(\tau))} < C. \quad (2.4)$$

We postpone the proof of this lemma to the appendix.

### 3 Positive singular solution in the Cone

We keep the assumptions and notations of the previous section, and we consider the cone

$$C_{\omega(\tau)} = \{(r, \theta) : r > 0, \theta \in \omega(\tau)\},$$

where  $r = |x|$  and  $\theta = \frac{x}{|x|}$ . We define the critical exponent

$$p^*(\tau) = \frac{n + \gamma(\tau)}{n + \gamma(\tau) - 2} \quad \text{with} \quad \gamma(\tau) = \frac{2-n}{2} + \sqrt{\left(\frac{n-2}{2}\right)^2 + \lambda(\tau)}.$$

We consider the problem

$$\begin{cases} \Delta_x u + u^p &= 0, & \text{in } C_{\omega(\tau)} \\ u &> 0, & \text{in } C_{\omega(\tau)} \\ u &= 0, & \text{on } \partial C_{\omega(\tau)} \setminus \{0\}. \end{cases} \quad (3.1)$$

If we set  $w = |x|^{-\frac{2}{p-1}} \phi(\theta)$ , we arrive at the problem

$$\begin{cases} \Delta_{\mathbb{S}^{n-1}} \phi - \frac{2}{p-1} \left(-\frac{2}{p-1} + n - 2\right) \phi + \phi^p &= 0, & \text{in } \omega(\tau) \\ \phi &= 0, & \text{on } \partial \omega(\tau). \end{cases} \quad (3.2)$$

By lemma 9 in [8], problem (3.2) has a positive solution  $\phi_p \in H_1(\omega(\tau)) \cap L^\infty(\omega(\tau))$  for any  $p \in (p^*, \infty)$  if  $n = 2$  or  $3$  and for any  $p \in (p^*(\tau), \frac{n+1}{n-3})$  if  $n \geq 4$ . Also as  $p \downarrow p^*(\tau)$  then  $-\frac{2}{p-1} \left(-\frac{2}{p-1} + n - 2\right) \uparrow \lambda(\tau)$  and

$$\phi_p = \left( \frac{\lambda - \frac{2}{p-1} \left(-\frac{2}{p-1} + n - 2\right)}{c_p} \right)^{\frac{1}{p-1}} (\phi_1 + o(1)),$$

where  $c_p = \int_{\omega(\tau)} \phi_1^{p+1} d\theta$ .

In addition, for the same range on  $p$ , by theorem 10 in [8], the function

$$w_p(\tau, r, \theta) = r^{-\frac{2}{p-1}} \phi_p(\tau, \theta)$$

is a positive solution of (3.1).

In the rest of this section, for convenience, we omit dependence on the parameter  $\tau$  writing  $\lambda = \lambda(\tau)$ ,  $\phi_1(\theta) = \phi_1(\tau, \theta)$  and so on.

Let  $p \in (p^*, \frac{n+2}{n-2})$ , we look for solutions of (3.1) of the form

$$u_1(x) = |x|^{-\frac{2}{p-1}} \phi(-\log |x|, \theta), \quad (3.3)$$

where  $\theta = \frac{x}{|x|}$ , so that the equation  $\Delta u + u^p = 0$  reads in terms of the function  $\phi$  defined for  $t \in \mathbb{R}$  and  $\theta \in \omega$ , as

$$\partial_t^2 \phi + A \phi_t - \varepsilon \phi + (\Delta_{\mathbb{S}^{n-1}} \phi + \lambda \phi) + \phi^p = 0, \quad (3.4)$$

where  $t = -\log r$ ,  $A = -\left(n - 2\frac{p+1}{p-1}\right)$  and  $\varepsilon = \lambda + \frac{2}{p-1}(n - \frac{2p}{p-1})$ .

Let  $\mu = \int_{\omega} \phi_1^{p+1} d\theta$ , we define  $a_\infty$  by

$$\mu a_\infty^{p-1} = \varepsilon.$$

We look for a positive function  $a$  which is a solution of

$$a''(t) + Aa'(t) - \varepsilon a(t) + \mu a^p(t) = 0, \quad (3.5)$$

which converges to 0 as  $t$  tends to  $-\infty$  and converges to  $a_\infty$  as  $t$  tends to  $+\infty$ . Observe that, when  $p \in (p^*, \frac{n+2}{n-2})$ , the coefficients  $A$  and  $\varepsilon$  are positive and, therefore, in this range, classical ODE techniques yield the existence of  $a$ , a positive heteroclinic solution of (3.5) tending to 0 at  $-\infty$  and tending to  $a_\infty$  at  $+\infty$ .

Observe that since the equation (3.5) is autonomous, the function  $a$  is not unique and  $a$  can be normalized so that  $a(0) = \frac{1}{2}a_\infty$ . For more informations about the function  $a$ , we refer the reader to lemmas 2.3, 2.4, 2.5 and appendix in [5].

**Proposition 3.1.** *Let  $0 \leq p_0 < \infty$  and  $\varepsilon$  be small enough, then there exists a unique operator*

$$G_{p_0} : a^{p_0} L^\infty(\mathbb{R} \times \omega) \mapsto a^{p_0} L^\infty(\mathbb{R} \times \omega),$$

*such that for any  $a^{-p_0} g \in L^\infty(\mathbb{R} \times \omega)$ , the function  $u = G_{p_0}(g)$  is the unique solution of*

$$L_p u = \left( \partial_t^2 + A \partial_t - \varepsilon + (\Delta_{\mathbb{S}^{n-1}} + \lambda) + p \phi_0^{p-1} \right) u = g; \quad \phi_0 = a(t) \phi_1(\theta),$$

*with zero Dirichlet boundary data.*

*Furthermore,*

$$\|d^{-1} a^{-p_0}(t) \psi\|_{L^\infty(\mathbb{R} \times \omega)} \leq \frac{C}{\varepsilon} \|a^{-p_0}(t) g\|_{L^\infty(\mathbb{R} \times \omega)}. \quad (3.6)$$

*If in addition  $g(t, \cdot)$  is  $L^2$ -orthogonal to  $\phi_1$  for a.e.  $t$ , then we have*

$$\|d^{-1} a^{-p_0}(t) \psi\|_{L^\infty(\mathbb{R} \times \omega)} \leq C \|a^{-p_0}(t) g\|_{L^\infty(\mathbb{R} \times \omega)}$$

*where  $d : \omega \rightarrow (0, \infty)$  denotes the distance function to  $\partial\omega$ .*

*Proof.* The proof follows the same lines as in lemma 2.6 in [5], so we will only focus on the differences. We first define  $\phi_*$  to be the positive solution of

$$\begin{cases} \Delta_{\mathbb{S}^{n-1}} \phi_* + \lambda \phi_* + \delta(\delta - n - 2\gamma + 2) \phi_* = & -1 & \text{in } \omega \\ \phi_* = & 0 & \text{on } \partial\omega \end{cases} \quad (3.7)$$

see the proof of lemma 2.6 in [5] with obvious modifications. Using the function  $(t, \theta) \rightarrow e^{-\delta t} \phi_*(\theta)$  as a barrier, as done in the paper [5], we can show that, given any function  $g$  such that  $a^{-p_0} g \in L^\infty(\mathbb{R} \times \omega)$  and given  $t_1 < -1 < 1 < t_2$ , we can solve the equation

$$L_p u = g$$

in  $(t_1, t_2) \times \omega$  with 0 boundary conditions.

To prove the estimate (3.6), we argue by contradiction, assuming that

$$\|a^{-p_0} \psi_i\|_{L^\infty} = 1$$

and

$$\lim_{i \rightarrow \infty} \|a^{-p_0} f_i\| = 0$$

we get a contradiction using similar argument as in lemma 2.6 in [5]. The rest of the proof is the same as in lemma 2.6 in [5] with obvious modifications so we omit it here.  $\square$

*Proof of theorem 1.1.* We look for a solution to problem (3.4) of the form

$$\phi = a(t)\phi_1(\theta) + \psi(t, \theta),$$

and we let  $G_p$  to be the operator defined in proposition 3.1. To conclude the proof, it is enough to find a function  $\psi$  solution of the fixed point problem

$$\psi = -G_p(\mathcal{M}(\phi_0) + \mathcal{Q}(\psi)),$$

where

$$\phi_0(t, \theta) = a(t)\phi_1(\theta),$$

$$\mathcal{M}(\phi_0) = a^p(\phi_1^p - \mu\phi_1)$$

$$\mathcal{Q}(\psi) = |\phi_0 + \psi|^p - \phi_0^p - p\phi_0^{p-1}\psi.$$

The rest of the proof is the same as in [5]. We recall here that  $\psi << a\phi_1$ . Also in [5], they have proven that if  $\varepsilon$  is small enough then there exists  $t_0$  such that for any  $t \leq -\frac{t_0}{\varepsilon}$ ,

$$\frac{1}{2}e^{\delta^-t} \leq a(t) \leq e^{\delta^-t},$$

with  $\delta^- = \frac{1}{2}(\sqrt{A^2 + 4\varepsilon} - A)$ . And the result follows, since

$$\frac{1}{2}(\sqrt{A^2 + 4\varepsilon} - A) + \frac{2}{p-1} = n + \gamma - 2.$$

□

### Remark 3.2.

If  $1 < p_0 < p$  is close enough to  $p$ , we can apply a fix point argument like in the proof of theorem 1.1, for the operator  $G_{p_0}$ .

In view of the proof of lemma 2.1,  $\phi_* = \phi_*(t, \cos(s\beta(\tau)))$ .

Thus if the function  $g$  in proposition 3.1 is of the form  $g = g(t, \cos(s\beta(\tau)))$ , we have that the solution  $u = G_{p_0}(g)$  is of the form  $u = u(t, \cos(s\beta(\tau)))$ . Hence we obtain, that the solution  $u_1$  in theorem 1.1 is of the form

$$u_1 = r^{-\frac{2}{p-1}}u_1(r, \cos(s\beta(\tau))).$$

### 3.1 Regularity of the solution $u_1$ with respect $\tau$

We first recall some definitions and known results, see the book of Gilbarg and Trudinger [7] for the proofs.

Let

$$Lu = a^{i,j}(x)D_{i,j}u + b^i(x)D_iu + c(x)u = g(x), \quad a^{i,j} = a^{j,i},$$

where the coefficients  $a^{i,j}$ ,  $b^i$ ,  $c$  and the function  $g$  are defined in an open bounded domain  $\Omega \subset \mathbb{R}^n$  and

$$a^{i,j}\xi_i\xi_j \leq \mu|\xi|^2; \quad \mu > 0.$$

We assume that

$$\|a^{i,j}\|_{C^{2,a}}, \|b^i\|_{C^{2,a}}, \|c\|_{C^{2,a}} \leq \Lambda.$$



**Definition 3.3.** We say that a bounded domain  $\Omega \subset \mathbb{R}^n$  and its boundary  $\partial\Omega$  are of class  $C^{k,a}$ ,  $0 \leq a \leq 1$ , if at each point  $x \in \partial\Omega$  there is a ball  $B_r(x)$  and a one-to-one mapping  $\psi$  from  $B_r(x)$  onto  $D \subset \mathbb{R}^n$  such that:

$$\psi(B_r(x) \cap \Omega) \subset \mathbb{R}_+^n, \quad \psi(B_r(x) \cap \partial\Omega) \subset \partial\mathbb{R}_+^n, \quad \psi \in C^{k,a}(B_r(x)) \text{ and } \psi^{-1} \in C^{k,a}(D).$$

A domain  $\Omega$  will be said to have a boundary portion  $T \subset \partial\Omega$  of class  $C^{k,a}$ , if at each point  $x \in T$  there is a ball  $B_r(x)$  in which the above conditions are satisfied and such that  $B_r(x) \cap \partial\Omega \subset T$ .

**Proposition 3.4. (Lemma 6.18 in [7]).** Let  $0 < a \leq 1$  and  $\Omega$  be a domain with a  $C^{2,a}$  boundary portion  $T$ , and let  $\phi \in C^{2,a}(\overline{\Omega})$ . Suppose that  $u$  is a  $C^2(\Omega) \cap C_0(\overline{\Omega})$  function satisfying  $Lu = g$  in  $\Omega$ ,  $u = \phi$  on  $T$ , where  $g$  and the coefficients of the strictly elliptic operator  $L$  belong to  $C^a(\overline{\Omega})$ . Then  $u \in C^{2,a}(\Omega \cup T)$ .

**Proposition 3.5. (Corollary 6.7 in [7]).** Let  $0 < a \leq 1$  and  $\Omega$  be a domain with a  $C^{2,a}$  boundary portion  $T$ , and let  $\phi \in C^{2,a}(\overline{\Omega})$ . Suppose that  $u$  is a  $C^{2,a}(\Omega \cup T)$  function satisfying  $Lu = g$  in  $\Omega$ ,  $u = \phi$  on  $T$ . Then, if  $x \in T$  and  $B = B_\rho(x)$  is a ball with radius  $\rho < \text{dist}(x, \partial\Omega - T)$ , we have

$$\|u\|_{C^{2,a}(B \cap \Omega)} \leq C(n, \mu, \Lambda, \Omega \cap B_\rho(x)) \left( \|u\|_{C(\Omega)} + \|\phi\|_{C^{2,a}(\overline{\Omega})} + \|g\|_{C^a(\Omega)} \right).$$

We first prove the following result

**Lemma 3.6.** Let  $\tau \in \mathbb{R}$  be fixed,  $x \in \mathbb{R}^n$ ,  $n \geq 2$ ,  $g \in C^a(\overline{C_\omega} \setminus \{0\})$  and  $u = G_p(g)$  be the operator in proposition 3.1. Then

$$\begin{aligned} |\nabla_x u(\tau, x)| &\leq C(n, p, \lambda, C_\omega(\tau), g) |x|^{-1} \\ |D_x^2 u(\tau, x)| &\leq C(n, p, \lambda, C_\omega(\tau), g) |x|^{-2}. \end{aligned} \quad (3.8)$$

*Proof.* First we note that  $\|u(\tau, \cdot)\|_{L^\infty(C_\omega(\tau))} \leq C\|g(t, \cdot)\|_{L^\infty(C_\omega(\tau))}$  and  $u$  is a solution of

$$\begin{cases} -\Delta_x u + \frac{4}{p-1} \frac{x \cdot \nabla_x u}{|x|^2} + \frac{2}{p-1} \left( n - \frac{2}{p-1} - 2 \right) \frac{u}{|x|^2} - p \frac{\phi_0^{p-1} u}{|x|^2} = -\frac{g}{|x|^2}, & \text{in } C_\omega(\tau) \\ u = 0 & \text{in } \partial C_\omega(\tau) \setminus \{0\}. \end{cases} \quad (3.9)$$

Set  $R = |x|$ , consider the domain

$$\Omega_R = \{y \in C_\omega : \frac{R}{4} < |y| < 4R\},$$

and let  $y = \frac{x}{R}$  and define  $v(y) = u(\tau, Ry)$ . Then  $y \in \Omega_1$  and  $v$  is a solution of

$$\begin{cases} -\Delta v + \frac{4}{p-1} \frac{y \cdot \nabla_y v}{|y|^2} + \frac{2}{p-1} \left( n - \frac{2}{p-1} - 2 \right) \frac{v}{|y|^2} - p \frac{\phi_0^{p-1} v}{|y|^2} = -\frac{g}{|y|^2}, & \text{in } \Omega_1 \\ v = 0 & \text{in } T, \end{cases} \quad (3.10)$$

where we have set

$$T = \partial\Omega_1 \setminus \{y \in C_\omega : |y| = \frac{1}{4} \text{ or } |y| = 4\}.$$

Let  $0 < \varepsilon < \frac{\rho}{4}$  be small enough, where  $\rho$  is the defined in proposition 3.5 with  $\Omega = \Omega_1$ . Let  $y_0 \in \partial\Omega_1 \setminus \{y \in C_\omega : |y| = \frac{1}{6} \text{ or } |y| = \frac{8}{3}\}$  then by propositions 3.4 and 3.5 we have

$$\|v\|_{C^2(B_\rho(\psi_0) \cap \Omega_2)} \leq C(n, \mu, \Lambda, \Omega_1 \cap B_\rho(y_0)) \|g\|_{C^a(\overline{\Omega_1})}$$

where in the last inequality we have used the estimate in proposition 3.1.

We note here that  $\rho$  depends only on  $\Omega_1$  and not on  $y_0$ . Thus if we apply a covering argument and standard interior Schauder estimates we have

$$\|v\|_{C^2(\Omega_{\frac{1}{2}})} \leq C(n, \mu, \Lambda, \Omega_1, \rho) \|g(x)\|_{C^a(\overline{\Omega_1})}.$$

Using the facts that  $x \in \Omega_{\frac{R}{2}}$ ,  $\nabla v(y) = R \nabla u(x)$ ,  $D_{i,j}v = R^2 D_{i,j}u$ ,  $R = |x|$  and the above estimate, the result follows at once.  $\square$

In the rest of this paper we assume that the Lipschitz spherical cap  $\omega(\tau)$  has the property:

*there exists  $\tilde{\varepsilon} > 0$ , such that for any  $p \in (\sup_{\tau \in \mathbb{R}} p^*(\tau), \sup_{\tau \in \mathbb{R}} p^*(\tau) + \tilde{\varepsilon})$ , there exists a solution  $u_1$  of theorem 1.1. Thus  $\varepsilon(\tau)$  is a smooth bounded function with bounded derivatives and there exist  $\varepsilon_0, \varepsilon_1 > 0$  such that  $\varepsilon_0 \leq \varepsilon(\tau) \leq \varepsilon_1$ ,  $\forall \tau \in \mathbb{R}$ .*

Now, we recall some facts from the proof of theorem 1.1. Let  $a(\tau, t)$  be the solution of the problem

$$\partial_t^2 a + A \partial_t a - \varepsilon(\tau) a + \mu(\tau) a^p = 0, \quad (3.11)$$

where  $A = -\left(n - 2\frac{p+1}{p-1}\right)$ ,  $\varepsilon(\tau) = \lambda(\tau) + \frac{2}{p-1}(n - \frac{2p}{p-1})$ ,  $\mu(\tau) = \int_{\omega(\tau)} \phi_1^{p+1}(\tau, \theta) d\theta$  and  $\mu(\tau) a_\infty^{p-1}(\tau) = \varepsilon(\tau)$ . Recall also that we have chosen  $a(\tau, t)$  such that

$$a(\tau, 0) = \frac{1}{2} a_\infty(\tau), \quad \lim_{t \rightarrow \infty} a(\tau, t) = a_\infty(\tau), \quad \text{and} \quad \lim_{t \rightarrow -\infty} a(\tau, t) = 0.$$

We next prove the following lemma

**Lemma 3.7.** *Let  $a$  be the solution of (3.11),  $\varepsilon_0 = \inf_{\tau \in \mathbb{R}} \varepsilon(\tau)$ ,*

$$\tilde{\delta}^+(\tau) = \frac{-A + \sqrt{A^2 - 4(p-1)\varepsilon(\tau)}}{2} \quad \text{and} \quad \delta^-(\tau) = \frac{-A + \sqrt{A^2 + 4\varepsilon(\tau)}}{2}.$$

*Then there exists  $\tilde{t} > 0$  such that*

$$\begin{aligned} \left| \frac{\partial a}{\partial \tau}(\tau, t) \right| &\leq C(\varepsilon_0, p, n) |t| e^{\delta^-(\tau)t}, & \forall (\tau, t) \in \mathbb{R} \times (-\infty, -\frac{\tilde{t}}{\varepsilon_0}), \\ \left| \frac{\partial^2 a}{\partial \tau^2}(\tau, t) \right| &\leq C(\varepsilon_0, p, n) |t|^2 e^{\delta^-(\tau)t}, & \forall (\tau, t) \in \mathbb{R} \times (-\infty, -\frac{\tilde{t}}{\varepsilon_0}), \\ \left| \frac{\partial a}{\partial \tau}(\tau, t) \right| &\leq C(\varepsilon_0, p, n) |t| e^{\tilde{\delta}^+(\tau)t}, & \forall (\tau, t) \in \mathbb{R} \times (\frac{\tilde{t}}{\varepsilon_0}, \infty), \\ \left| \frac{\partial^2 a}{\partial \tau^2}(\tau, t) \right| &\leq C(\varepsilon_0, p, n) |t|^2 e^{\tilde{\delta}^+(\tau)t}, & \forall (\tau, t) \in \mathbb{R} \times (\frac{\tilde{t}}{\varepsilon_0}, \infty). \end{aligned}$$

*And*

$$\begin{aligned} \left| \frac{\partial a}{\partial \tau}(\tau, t) \right| &\leq C(\varepsilon_0, p, n), & \forall (\tau, t) \in \mathbb{R} \times [-\frac{\tilde{t}}{\varepsilon_0}, \frac{\tilde{t}}{\varepsilon_0}], \\ \left| \frac{\partial^2 a}{\partial \tau^2}(\tau, t) \right| &\leq C(\varepsilon_0, p, n), & \forall (\tau, t) \in \mathbb{R} \times [-\frac{\tilde{t}}{\varepsilon_0}, \frac{\tilde{t}}{\varepsilon_0}]. \end{aligned}$$

*Proof.* By our assumptions and lemma 2.5 in [5] there exists a constant  $\bar{t} < 0$  (independent on  $p$ ,  $\mu$  and  $\tau$ ) such that

$$\frac{1}{2}e^{\delta^-(\tau)t} \leq \frac{a(\tau, t)}{a_\infty(\tau)} \leq e^{\delta^-(\tau)t}, \quad \forall t \leq \frac{\bar{t}}{\varepsilon_0},$$

where

$$\delta^-(\tau) = \frac{-A + \sqrt{A^2 + 4\varepsilon(\tau)}}{2}.$$

Choose  $\tau_0 \in \mathbb{R}$  and set  $a(\tau, t) = a_\infty(\tau)(e^{\delta^-(\tau)t} + w(\tau, t))$ . Then  $w$  is a solution of the fixed point problem

$$\begin{aligned} w &= -\varepsilon e^{\delta^-(\tau)t} \int_{-\infty}^t e^{-2\delta^-(\tau)\zeta - A\zeta} \left( \int_{-\infty}^{\zeta} e^{\delta^-(\tau)s + As} \left( e^{\delta^-(\tau)s} + w \right)^p ds \right) d\zeta \\ &:= T[w]. \end{aligned} \tag{3.12}$$

Indeed, let  $1 < p_0 < p$  and  $\rho$  be sufficiently small such that for any  $\tau \in O_{\tau_0} = \{\tau \in \mathbb{R} : |\tau - \tau_0| < \rho\}$  we have

$$p\delta^-(\tau) \geq p_0\delta^-(\tau_0) \quad \text{and} \quad p\delta^-(\tau_0) \geq p_0\delta^-(\tau).$$

Thus, it is easy to find a fixed point in the set of functions defined in  $(-\infty, \frac{\bar{t}}{\varepsilon_0})$  and satisfying

$$|w| \leq \frac{1}{2}e^{p_0\delta^-(\tau_0)t}$$

provided  $|\bar{t}|$  is fixed large enough (independent of  $p$  and  $\tau$ ).

Now let

$$G = \{g : (-\infty, \frac{\bar{t}}{\varepsilon_0}) \mapsto \mathbb{R} : \|e^{-p_0\delta^-(\tau_0)t}g\|_{L^\infty(-\infty, \frac{\bar{t}}{\varepsilon_0})} < C\}$$

and define  $F(\tau, g) = g - T(g)$ . By (3.12) we can apply the Implicit Function theorem in the domain  $O_{\tau_0} \times G$  to obtain that there exists a unique function  $w$  such that

$$F(\tau, w(\tau, t)) = 0 \quad \text{for any} \quad |\tau - \tau_0| < \rho_0 < \rho$$

for some  $\rho_0$  small enough. On the other hand since  $T(g)$  is smooth with respect  $\tau$  we have that  $w(\tau, t)$  is smooth with respect  $\tau$ .

Notice that

$$0 = F_\tau(\tau, w(\tau, t)) + F_g(\tau, w(\tau, t)) \frac{\partial w}{\partial \tau}$$

thus we have

$$\left| \frac{\partial w}{\partial \tau}(\tau, t) \right| \leq C(\varepsilon_0, p, n) |t| e^{\delta^-(\tau)t}, \tag{3.13}$$

provided  $|\bar{t}|$  is fixed large enough. Similarly we have

$$\left| \frac{\partial^2 w}{\partial \tau^2}(\tau, t) \right| \leq C(\varepsilon_0, p, n) |t|^2 e^{\delta^-(\tau)t}. \tag{3.14}$$

By (3.12) and the above inequalities we have that the derivatives  $\frac{\partial^2 w}{\partial \tau \partial t}$ ,  $\frac{\partial^3 w}{\partial^2 \tau \partial t}$  exist and are bounded.

Since the choice of  $\tau_0$  is abstract, we conclude that the functions  $a$ ,  $\partial_t a \in C^2$  with respect  $\tau$ , for any  $t \leq \frac{\bar{t}}{\varepsilon_0}$ . We also have

$$\begin{aligned} \left| \frac{\partial a}{\partial \tau}(\tau, t) \right| &\leq C(\varepsilon_0, p, n) |t| e^{\delta^-(\tau)t}, \quad \forall (\tau, t) \in \mathbb{R} \times (-\infty, -\frac{\bar{t}}{\varepsilon_0}), \\ \left| \frac{\partial^2 a}{\partial \tau^2}(\tau, t) \right| &\leq C(\varepsilon_0, p, n) |t|^2 e^{\delta^-(\tau)t}, \quad \forall (\tau, t) \in \mathbb{R} \times (-\infty, -\frac{\bar{t}}{\varepsilon_0}). \end{aligned} \tag{3.15}$$

Let  $t_0 \in \left(-\infty, \frac{t}{\varepsilon_0}\right)$  such that  $a(\tau, t_0), \frac{\partial a(\tau, t_0)}{\partial t} \in C^2$  with respect  $\tau$ . Using standard ODE techniques we can prove that, if  $|h|$  is sufficiently small then

$$|a(\tau, t) - a(\tau + h, t)| \leq C(t)h, \quad \forall t \in \mathbb{R}, \quad (3.16)$$

where  $C(t)$  is a positive smooth function such that  $\lim_{t \rightarrow \infty} C(t) = \infty$ .

Choose  $|h|$  sufficiently small and set  $v_h = \frac{a(\tau+h, t) - a(\tau, t)}{h}$  and  $a(\tau) = a(\tau, t)$ . Then  $v_h$  satisfies

$$\begin{aligned} \frac{\partial^2 v_h}{\partial t^2} + A \frac{\partial v_h}{\partial t} - \varepsilon(\tau + h)v_h &= -\mu(\tau + h) \frac{a^p(\tau + h) - a^p(\tau)}{h} \\ &\quad - \frac{\mu(\tau + h) - \mu(\tau)}{h} a^p(\tau) + \frac{\varepsilon(\tau + h) - \varepsilon(\tau)}{h} a(\tau), \quad \text{in } (t_0, \infty), \\ v_h(\tau, t_0) &= \frac{a(\tau + h, t_0) - a(\tau, t_0)}{h}, \\ \frac{\partial v_h(\tau, t_0)}{\partial t} &= \frac{\frac{\partial a(\tau+h, t_0)}{\partial t} - \frac{\partial a(\tau, t_0)}{\partial t}}{h}. \end{aligned} \quad (3.17)$$

Using the following expansion

$$\begin{aligned} a^p(\tau + h) &= a^p(\tau) + p a^{p-1}(\tau, t) (a(\tau + h) - a(\tau)) \\ &\quad + \frac{1}{2} \int_{a(\tau)}^{a(\tau+h)} p(p-1)t^{p-2}(a(\tau + h) - t)dt, \end{aligned}$$

thus by the properties of initial data in (3.17), our assumptions on  $\mu, \varepsilon$ , (3.16) and above equality, we can obtain by using standard ODE techniques in (3.17) that

$$|v_h|, \quad \left| \frac{\partial v_h}{\partial t} \right| < C(t),$$

where  $C(t)$  is a positive smooth function such that  $\lim_{t \rightarrow \infty} C(t) = \infty$ . Thus by Arzela Ascoli theorem, there exist a subsequence  $\{v_{h_n}\}$  such that  $v_{h_n} \rightarrow v$  locally uniformly and  $v$  satisfies

$$\begin{aligned} \frac{\partial^2 v}{\partial t^2} + A \frac{\partial v}{\partial t} - \varepsilon(\tau)v &= -\mu(\tau) p a^{p-1}(\tau, t)v - \mu'(\tau) a^p(\tau) + \varepsilon'(\tau) a(\tau) \quad \text{in } (t_0, \infty) \\ v(\tau, t_0) &= \frac{\partial a(\tau, t_0)}{\partial \tau} \\ \frac{\partial v(\tau, t_0)}{\partial t} &= \frac{\partial^2 a(\tau, t_0)}{\partial \tau \partial t}. \end{aligned}$$

By uniqueness of the above problem, we have that  $\lim_{h \rightarrow 0} v_h = v$  for all  $\tau \in \mathbb{R}$  and  $t \geq t_0$ . And thus  $\frac{\partial}{\partial \tau} a(\tau, t)$  exists for any  $(\tau, t) \in \mathbb{R}^2$ . Applying the same argument we can obtain also that  $\frac{\partial^2}{\partial \tau^2} a(\tau, t)$  exists for any  $(\tau, t) \in \mathbb{R}^2$ . The only difference is that we should use the fact that  $a(\tau, t) > c > 0$  for any  $(\tau, t) \in \mathbb{R} \times (t_0, \infty)$ .

Set  $a = a_\infty w$  then  $w$  satisfies

$$\partial_t^2 w + A \partial_t w - \varepsilon(\tau)w + \varepsilon(\tau)w^p = 0. \quad (3.18)$$

Let us now recall some facts from lemma 2.5 in [5]. Set

$$\tilde{\delta}^+(\tau) = \frac{-A + \sqrt{A^2 - 4(p-1)\varepsilon(\tau)}}{2} \quad \text{and} \quad \tilde{\delta}^-(\tau) = \frac{-A - \sqrt{A^2 - 4(p-1)\varepsilon(\tau)}}{2}.$$

There exists a  $\hat{t} > 0$  (independent on  $p$  and  $\tau$ ) such that ,  $\forall t \geq \frac{\hat{t}}{\varepsilon_0}$

$$\begin{aligned} \frac{1}{2}e^{\tilde{\delta}^-(\tau)t} &\leq 1 - w(\tau, t) \leq 2e^{\tilde{\delta}^-(\tau)t} \\ \frac{1}{C(\varepsilon_0)}w(1-w) &\leq \frac{\partial w}{\partial t} \leq C(\varepsilon_0)w(1-w). \end{aligned} \quad (3.19)$$

Notioce that the function  $\frac{\partial w}{\partial \tau}$  is a solution of

$$\frac{\partial^2 v}{\partial t^2} + A \frac{\partial v}{\partial t} - \varepsilon(\tau)v + pw^{p-1}(\tau, t)v = \varepsilon'(\tau)w^p(\tau) + \varepsilon'(\tau)w(\tau), \quad (3.20)$$

but the function  $\frac{\partial a}{\partial t}$  is one solution of the corresponding homogeneous problem. For the other solution of the homogeneous problem  $\psi$  we can easily prove by using (3.19) that

$$|\psi(t, \tau)| \leq C(\varepsilon_0)e^{\tilde{\delta}^-(\tau)t}.$$

Thus by the representation formula and the properties of  $w$ , we can easily get

$$\left| \frac{\partial w}{\partial \tau} \right| \leq C(\varepsilon_0, p, n)|t|e^{\tilde{\delta}^+(\tau)t}, \quad \forall t \geq \frac{\tilde{t}}{\varepsilon_0}.$$

Using the estimates (3.19) and the fact that  $w$  is a solution of (3.18), we can prove that

$$\left| \frac{\partial^2 w}{\partial t^2} \right| < C(\varepsilon_0, n, p)e^{\tilde{\delta}^+(\tau)t}.$$

Setting  $w = \left(1 - e^{\tilde{\delta}^+(\tau)t} + v\right)$ , then  $v$  can be written (see appendix in [5])

$$\begin{aligned} v &= \varepsilon e^{\tilde{\delta}^-(\tau)t} \int_{t_p}^t e^{-2\tilde{\delta}^-(\tau)\zeta - A\zeta} \left( \int_{\zeta}^{\infty} e^{\tilde{\delta}^-(\tau)s + As} \mathcal{Q} \left( -e^{\tilde{\delta}^+(\tau)s} + v \right) ds \right) d\zeta \\ &\quad + \lambda_p e^{\tilde{\delta}^-(\tau)t}. \end{aligned} \quad (3.21)$$

where  $Q(x) = |1 + x|^p - 1 - px$ ,  $t_p$  is large enough and  $\lambda_p(\tau)$  is a smooth bounded function. Thus by (3.21) and the definition of  $v$  we can prove that there exists a constant  $C > 0$  such that

$$\frac{1}{C}e^{\tilde{\delta}^+(\tau)t} \leq -\partial_t^2 w(\tau, t) \leq Ce^{\tilde{\delta}^+(\tau)t}, \quad \forall t \geq t_p.$$

By the same argument we can prove that

$$\left| \frac{\partial^2 w}{\partial \tau^2}(\tau, t) \right| \leq C(\varepsilon_0, p, n)|t|^2 e^{\tilde{\delta}^+(\tau)t}, \quad \forall t \geq \frac{\tilde{t}}{\varepsilon_0}.$$

This ended the proof. □

**Lemma 3.8.** *Let  $u_1$  be the solution given by theorem 1.1, then the following estimates hold*

$$|\partial_{\tau} u_1(\tau, x)| \leq C|x|^{-\frac{2}{p-1}} \quad \text{and} \quad |\partial_{\tau}^2 u_1(\tau, x)| \leq C|x|^{-\frac{2}{p-1}},$$

where the constant  $C$  does not depend on  $\tau$  and  $x$ .

*Proof.* In view of the proof of theorem 1.1,

$$u_1 = |x|^{-\frac{2}{p-1}} f(\tau, \theta) = |x|^{-\frac{2}{p-1}} (a(\tau, t)\phi_1(\tau, \theta) + \psi(\tau, \theta)),$$

where  $\psi$  is a solution of the fixed point problem

$$\psi = -G_p(\mathcal{M}(\phi_0) + \mathcal{Q}(\psi)), \quad (3.22)$$

where  $\phi_0(\tau, \theta) = a(\tau, t)\phi_1(\tau, \theta)$ ,  $\mathcal{M}(\phi_0) = a^p(\phi_1^p - \mu\phi_1)$  and

$$\mathcal{Q}(\psi) = |\phi_0 + \psi|^p - \phi_0^p - p\phi_0^{p-1}\psi.$$

We recall here that  $|\psi(t, \theta)| \ll a(\tau, t)\phi_1(\tau, \theta)$ .

Here we will only treat the case  $n \geq 3$ . For  $n = 2$  the proof is the same.

By uniqueness, our assumptions on  $\omega(\tau)$ , and remark 3.2.  $\psi = \psi(t, \tilde{s})$ ,  $\tilde{s} \in (0, \beta(\tau))$ ,  $\theta_1 = \cos \tilde{s}$ , where  $\beta(\tau)$  is a positive smooth function such that

$$0 < \inf_{\tau \in \mathbb{R}} \beta(\tau) \leq \sup_{\tau \in \mathbb{R}} \beta(\tau) < \pi.$$

Then  $\psi$  satisfies

$$\begin{aligned} & (\partial_t^2 + A\partial_t - \varepsilon(\tau)) \psi + \sin^{2-n}(\tilde{s})\partial_{\tilde{s}}(\sin^{n-2}(\tilde{s})\partial_{\tilde{s}}\psi) + \lambda(\tau)\psi + p\phi_0^{p-1}\psi \\ & = -\mathcal{M}(\phi_0) - \mathcal{Q}(\psi), \end{aligned}$$

for any  $(t, \tilde{s}) \in \mathbb{R} \times (0, \beta(\tau))$ , and  $\psi(t, \beta(\tau)) = 0$ .

Setting now  $s = \frac{\tilde{s}}{\beta(\tau)}$ , we have that  $\psi(\tau, t, s)$  satisfies

$$\begin{aligned} \tilde{L}_p\psi &:= (\partial_t^2 + A\partial_t - \varepsilon(\tau)) \psi + \frac{1}{\beta^2(\tau)}\partial_s^2\psi \\ &+ (n-2)\frac{\cos(\beta(\tau)s)}{\beta(\tau)\sin(\beta(\tau)s)}\partial_s\psi + \lambda\psi + p\phi_0^{p-1}\psi = -\mathcal{M}(\phi_0) - \mathcal{Q}(\psi), \end{aligned} \quad (3.23)$$

for any  $(t, s) \in \mathbb{R} \times (0, 1)$ , and  $\psi(\tau, t, 1) = 0$ .

Let  $1 < p_0 < p$  such that  $p - p_0$  is small enough and let  $g : \mathbb{R} \times (0, 1) \rightarrow \mathbb{R}$  such that  $g \in C^a(\mathbb{R} \times [0, 1])$  for some  $0 < a \leq 1$ , and

$$\sup_{\tau \in \mathbb{R}} \sup_{(t, s) \in \mathbb{R} \times (0, 1)} |a^{-p}(\tau, t)g(t, s)| < \infty.$$

Let  $u(\tau, t, s) = -\tilde{G}_p(\mathcal{M}(\phi_0) + \mathcal{Q}(g))$  be the solution of (3.23). This solution exists since problem (3.23) is equivalent to (3.22). In addition, by proposition 3.1 we have the following estimate

$$\begin{aligned} \sup_{(t, s) \in \mathbb{R} \times (0, 1)} |d^{-1}a^{-p_0}(\tau, t)u(\tau, t)| &\leq C \sup_{(t, s) \in \mathbb{R} \times (0, 1)} |a^{-p_0}(\tau, t)\mathcal{M}(\phi_0)(\tau, t, s)| \\ &+ \frac{C}{\varepsilon} \sup_{(t, s) \in \mathbb{R} \times (0, 1)} |a^{-p_0}(\tau, t)\mathcal{Q}(g)(\tau, t)|, \end{aligned} \quad (3.24)$$

for some constant  $C > 0$  which does not depend on  $\tau$ .

We can easily prove that

$$\lim_{h \rightarrow 0} \sup_{(t, s) \in \mathbb{R} \times (0, 1)} |u(\tau + h, t, s) - u(\tau, t, s)| = 0.$$

Recall the definitions

$$u_h(\tau, t, s) = \frac{u(\tau + h, t, s) - u(\tau, t, s)}{h}, \quad u(\tau) = u(\tau, t, s), \dots$$

Clearly  $u_h$  satisfies

$$\begin{aligned}
& (\partial_t^2 + A\partial_t - \varepsilon(\tau + h)) u_h(\tau) + \frac{1}{\beta^2(\tau + h)} \partial_s^2 u_h(\tau) \\
& + \frac{(n-2) \cos(\beta(\tau + h)s)}{\beta(\tau + h) \sin(\beta(\tau + h)s)} \partial_s u_h(\tau) + \lambda(\tau + h) u_h + p \phi_0^{p-1}(\tau + h) u_h(\tau) \\
& = -\frac{\frac{1}{\beta^2(\tau + h)} - \frac{1}{\beta^2(\tau)}}{h} \partial_s^2 u(\tau) + \frac{\varepsilon(\tau + h) - \varepsilon(\tau)}{h} u(\tau) - \frac{\lambda(\tau + h) - \lambda(\tau)}{h} u(\tau) \\
& - (n-2) \frac{\frac{\cos(\beta(\tau + h)s)}{\beta(\tau + h) \sin(\beta(\tau + h)s)} - \frac{\cos(\beta(\tau)s)}{\beta(\tau) \sin(\beta(\tau)s)}}{h} \partial_s u(\tau) - p \frac{\phi_0^{p-1}(\tau + h) - \phi_0^{p-1}(\tau)}{h} u(\tau) \\
& - \frac{\mathcal{M}(\phi_0)(\tau + h) - \mathcal{M}(\phi_0)(\tau)}{h} - \frac{\mathcal{Q}(g)(\tau + h) - \mathcal{Q}(g)(\tau)}{h}.
\end{aligned}$$

Now notice that  $u(\tau, t, s) = w(t, \cos(s\beta(\tau))) = v(\tau, x)$ , where  $x_1 = |x| \cos(s\beta(\tau))$ . In addition,  $v(\tau, x)$  satisfies

$$\begin{cases} -\Delta_x v + \frac{4}{p-1} \frac{x \cdot \nabla_x v}{|x|^2} + \frac{2}{p-1} \left( n - \frac{2}{p-1} - 2 \right) \frac{v}{|x|^2} - p \frac{\phi_0^{p-1} v}{|x|^2} = -\frac{g}{|x|^2}, & \text{in } C_\omega(\tau) \\ v = 0 & \text{in } \partial C_\omega(\tau) \setminus \{0\}. \end{cases}$$

Thus by lemma 3.6 we have

$$\left| \frac{1}{\sin s\beta(\tau)} \frac{\partial u}{\partial s} \right| \leq \frac{1}{\inf_{\tau \in \mathbb{R}} \beta(\tau)} |x| |v_{x_1}| < C.$$

Similarly we can obtain  $\left| \frac{\partial^2 u}{\partial s^2} \right| < C$  for some constant  $C > 0$  which does not depend on  $\tau$ .

Thus we have

$$\begin{aligned}
& \sup_{(t,s) \in \mathbb{R} \times (0,1)} \left| \frac{1}{\sin s\beta(\tau)} \frac{\partial u}{\partial s}(\tau, t, s) \right| < C \\
& \sup_{(t,s) \in \mathbb{R} \times (0,1)} \left| \frac{\partial^2 u}{\partial s^2}(\tau, t, s) \right| < C,
\end{aligned} \tag{3.25}$$

where the constant  $C > 0$  does not depend on  $\tau$ . Now we have

$$\begin{aligned}
& \limsup_{h \rightarrow 0} \sup_{\tau \in \mathbb{R}} \left| \frac{\frac{\cos(\beta(\tau + h)s)}{\beta(\tau + h) \sin(\beta(\tau + h)s)} - \frac{\cos(\beta(\tau)s)}{\beta(\tau) \sin(\beta(\tau)s)}}{h} \partial_s u(\tau) \right| \\
& = \sup_{\tau \in \mathbb{R}} \left| \left( -\frac{\beta'(\tau)}{\beta^2(\tau)} \cot(\beta(\tau)s) - \frac{s\beta'(\tau)}{\sin^2 \beta(\tau)s} \right) \partial_s u(\tau) \right| < C,
\end{aligned}$$

where in the last inequality we have used the fact that

$$0 < \inf_{\tau \in \mathbb{R}} \beta(\tau) \leq \sup_{\tau \in \mathbb{R}} \beta(\tau) < \pi$$

and (3.25). Using the fact that

$$\begin{aligned}
& a^p(\tau + h) \phi_1^p(\tau + h) - a^p(\tau) \phi_1^p(\tau) \\
& = (a^p(\tau + h) - a^p(\tau)) \phi_1^p(\tau + h) + a^p(\tau) (\phi_1^p(\tau + h) - \phi_1^p(\tau)),
\end{aligned}$$

and

$$\begin{aligned} a^p(\tau + h) &= a^p(\tau) + pa^{p-1}(\tau)(a^p(\tau + h) - a^p(\tau)) \\ &\quad + \frac{p(p-1)}{2} \int_{a^p(\tau)}^{a^p(\tau+h)} t^{p-2}(a^p(\tau + h) - t)dt, \end{aligned}$$

(the same for  $\phi_1$ ), and lemmas 2.1, 3.7, we have that

$$\left| \lim_{h \rightarrow 0} \frac{\mathcal{M}(\phi_0)(\tau + h) - \mathcal{M}(\phi_0)(\tau)}{h} \right| = \left| \frac{\partial \mathcal{M}(\phi_0)}{\partial \tau} \right| < C.$$

Similarly we have that

$$\left| \lim_{h \rightarrow 0} \frac{\mathcal{Q}(g)(\tau + h) - \mathcal{Q}(g)(\tau)}{h} \right| = \left| \frac{\partial \mathcal{Q}(g)}{\partial \tau} \right| < C.$$

By proposition 3.1 we have

$$\sup_{\tau \in \mathbb{R}} \sup_{(t,s) \in \mathbb{R} \times (0,1)} |u_h| < C$$

and thus by Arzela Ascoli theorem, there exist a subsequence  $\{u_{h_n}\}$  such that  $u_{h_n} \rightarrow v$  locally uniformly and  $v(\tau, t, s)$  satisfies

$$\begin{aligned} &(\partial_t^2 + A\partial_t - \varepsilon(\tau))v + \frac{1}{\beta^2(\tau)}\partial_s^2 v + \frac{\cos(\beta(\tau)s)}{\beta(\tau)\sin(\beta(\tau)s)}\partial_s v \\ &+ \lambda(\tau)u + p\phi_0^{p-1}(\tau)v = H(\phi_1, a, g), \end{aligned}$$

with  $v(\tau, t, 1) = 0$ . Notice that

$$\sup_{\tau \in \mathbb{R}} \sup_{(t,s) \in \mathbb{R} \times (0,1)} |H(\tau, t, s)| < C,$$

thus by proposition 3.1  $v$  is a unique solution. Furthermore,

$$\lim_{h \rightarrow 0} u_h = v = \frac{\partial u}{\partial \tau},$$

and

$$\sup_{\tau \in \mathbb{R}} \sup_{(t,s) \in \mathbb{R} \times (0,1)} \left| \frac{\partial u}{\partial \tau}(\tau, s, t) \right| < C, \quad (3.26)$$

for some constant  $C$  independent on  $g$ .

Similarly as (3.25) we can prove,

$$\begin{aligned} &\sup_{\tau \in \mathbb{R}} \sup_{(t,s) \in \mathbb{R} \times (0,1)} \left| \frac{1}{\sin s \beta(\tau)} \frac{\partial^2 u}{\partial \tau \partial s}(\tau, t, s) \right| < C \\ &\sup_{\tau \in \mathbb{R}} \sup_{(t,s) \in \mathbb{R} \times (0,1)} \left| \frac{\partial^3 u}{\partial \tau \partial s \partial s}(\tau, t, s) \right| < C \end{aligned}$$

and by the same argument as above

$$\sup_{\tau \in \mathbb{R}} \sup_{(t,s) \in \mathbb{R} \times (0,1)} \left| \frac{\partial^2 u}{\partial \tau \partial \tau}(\tau, t, s) \right| < C, \quad (3.27)$$

where  $C$  is a constant which depends on  $g$ .



Now we consider the fix point problem (3.23). Let  $\tau_0 \in \mathbb{R}$  and  $\rho$  be small enough such that for any  $\tau \in O_{\tau_0} = \{\tau \in \mathbb{R} : |\tau - \tau_0| < \rho\}$  we have  $p\delta^-(\tau) \geq p_0\delta^-(\tau_0)$ , where

$$\delta^-(\tau) = \frac{-A + \sqrt{A^2 + 4\varepsilon(\tau)}}{2}.$$

We can easily show that  $a^p(\tau, t) \leq Ca^{p_0}(\tau_0, t)$ ,  $\forall \tau \in O_{\tau_0}$ , for some positive constant  $C$  independent on  $\tau$  and  $t$ .

Now since  $0 < p - p_0$  is small enough, we can use a fix point argument like in [5] (see remark 3.2) in the Banach space

$$\mathbf{X} = \{g \in L^\infty(\mathbb{R} \times (0, 1)) : \sup_{(t,s) \in \mathbb{R} \times (0,1)} |a^{-p_0}(\tau_0, t)g(t, s)| < \infty\}$$

to prove that there exists a unique solution

$$\psi(\tau, t, s) = -\tilde{G}_p(\mathcal{M}(\phi_0) + \mathcal{Q}(\psi(\tau, t, s))), \quad \forall \tau \in O_{\tau_0}.$$

Now, let  $(\tau, g) \in O_{\tau_0} \times \mathbf{X}$ , we set the bounded operator

$$T(\tau, g) = g + \tilde{G}_p(g),$$

We can apply the Implicit Function theorem to  $O_{\tau_0} \times \mathbf{X}$  to obtain that:

let  $0 < \rho_0 \leq \rho$  be small enough, then for any  $\tau \in \{\tau \in \mathbb{R} : |\tau - \tau_0| < \rho_0\} \subset O_{\tau_0}$  there exists a function  $\psi(\tau, t, s)$  such that

$$T(\tau, \psi(\tau, t, s)) = 0.$$

Using (3.26), (3.27) and again the Implicit Function theorem, we can also prove that  $\partial_\tau \psi$ ,  $\partial_\tau^2 \psi$  exist. Furthermore using the fact that

$$0 = T_\tau(\tau, \psi(\tau)) + T_g(\tau, \psi(\tau))\partial_\tau \psi,$$

and the estimate (3.26) we have that

$$\sup_{\tau \in (\tau_0 - \rho_0, \tau_0 + \rho_0)} \sup_{(t,s) \in \mathbb{R} \times (0,1)} \left| \frac{\partial u}{\partial \tau}(\tau, t, s) \right| < C.$$

Similarly we have

$$\sup_{\tau \in (\tau_0 - \rho_0, \tau_0 + \rho_0)} \sup_{(t,s) \in \mathbb{R} \times (0,1)} \left| \frac{\partial^2 u}{\partial \tau \partial \tau}(\tau, t, s) \right| < C.$$

And the result follows since  $\tau_0$  is abstract. □

## 4 The proof of theorems 1.2 and 1.3

Let  $x \in \mathbb{R}^n$ ,  $n \geq 2$ ,  $R > 0$ ,  $B_R(0) \subset \mathbb{R}^n$  and

$$r_{\sigma(\tau)} = |x - \sigma(\tau)|,$$

where  $\sigma : \mathbb{R} \rightarrow \mathbb{R}^n$  is a smooth curve such that

$$\sup_{\tau \in \mathbb{R}} \{|\sigma(\tau)| + |\sigma'(\tau)| + |\sigma''(\tau)|\} < C < \infty.$$

Define

$$\hat{r}^2 = \sum_{i=1}^n |(x_i - \sup |\sigma(\tau)|)|^2 :$$

Given  $\tau$ , let  $(r_{\sigma(\tau)}, \theta) \in [0, \infty) \times \mathbb{S}^{n-1}$  be the spherical-coordinates of  $x \in \mathbb{R}^n$  centered at  $\sigma(\tau)$  abbreviated by  $x = (r_{\sigma(\tau)}, \theta)$ . We define the cone

$$\tilde{C}_{\omega(\tau)} = \{x = (r_{\sigma(\tau)}, \theta) : r_{\sigma(\tau)} > 0, \theta \in \omega(\tau)\} \subset \mathbb{R}^n.$$

and we denote by

$$\Omega_{\tau_1, \tau_2} = \{(\tau, x) \in (\tau_1, \tau_2) \times \mathbb{R}^n : x \in \tilde{C}_{\omega(\tau)}\} \subset \mathbb{R}^{n+1}.$$

$$\Omega_{\tau_1, \tau_2}^R = \Omega_{\tau_1, \tau_2} \cap \{(\tau, x) \in (\tau_1, \tau_2) \times \mathbb{R}^n : x \in B_R(\sigma(\tau))\} \subset \mathbb{R}^{n+1},$$

and

$$S_{\tau_1, \tau_2} = \{(\tau, x) \in [\tau_1, \tau_2] \times \mathbb{R}^n : r_{\sigma(\tau)} = 0\}.$$

Let  $C_{\delta, \rho}(\Omega_{\tau_1, \tau_2}^R)$  be the set of continuous function  $f \in C(\Omega_{\tau_1, \tau_2}^R)$  with norm

$$\|f\|_{C_{\delta, \rho}(\Omega_{\tau_1, \tau_2}^R)} := \sup_{(\tau, x) \in \Omega_{\tau_1, \tau_2}^R} \left( \chi_{[0, 1]}(r_{\sigma(\tau)}) r_{\sigma(\tau)}^{-\delta} |f| + \chi_{[1, \infty)}(r_{\sigma(\tau)}) \tilde{r}^{-\rho} |f| \right).$$

Let  $\delta \in (-n - \gamma + 2, \gamma)$ , we define  $\phi_\delta(\tau, \theta)$  to be the unique positive solution of

$$\begin{cases} \Delta_{\mathbb{S}^{n-1}} \phi_\delta + \lambda \phi_\delta + (\delta(\delta + n - 2) - \lambda) \phi_\delta &= -1, & \text{in } \omega(\tau) \\ \phi_\delta &= 0, & \text{on } \partial\omega(\tau). \end{cases}$$

Notice here that  $\lambda = \gamma^2 + \gamma(n - 2)$ , thus  $\delta(\delta + n - 2) - \lambda < 0$  if and only if  $\delta \in (-n - \gamma + 2, \gamma)$ . A direct computation shows that

$$-\Delta_x (|x|^\delta \phi_\delta) = |x|^{\delta-2}.$$

In view of lemma 2.1 we have that  $\phi_\delta = \phi_\delta(t)$  where  $t \in (0, \beta(\tau))$  and it satisfies

$$\begin{cases} \sin^{2-n} t \frac{d}{dt} \left( \sin^{n-2} t \frac{d\phi_\delta}{dt} \right) + \lambda \phi_\delta + (\delta(\delta + n - 2) - \lambda) \phi_\delta &= -1 & \text{in } (0, \beta(\tau)) \\ \phi_\delta(\beta(\tau)) &= 0. \end{cases}$$

We next set  $\beta^* = \sup_{\tau \in \mathbb{R}} \beta(\tau)$ , and  $\lambda^* = \inf_{\tau \in \mathbb{R}} \lambda(\tau)$ ,  $\gamma^* = \inf_{\tau \in \mathbb{R}} \gamma(\tau)$  and we let  $\phi_\delta^*$  be the solution of

$$\begin{cases} \sin^{2-n} t \frac{d}{dt} \left( \sin^{n-2} t \frac{d\phi_\delta^*}{dt} \right) + \lambda^* \phi_\delta^* + (\delta(\delta + n - 2) - \lambda^*) \phi_\delta^* &= -1 & \text{in } (0, \beta^*) \\ \phi_\delta(\beta^*) &= 0 \end{cases}$$

with  $\gamma \in (-n - \gamma^* + 2, \gamma^*)$ .

Thus  $\phi_\delta^*$  is the unique solution of the problem

$$\begin{cases} \Delta_{\mathbb{S}^{n-1}} \phi_\delta^* + \lambda^* \phi_\delta^* + (\delta(\delta + n - 2) - \lambda^*) \phi_\delta^* &= -1, & \text{in } \omega^* \\ \phi_\delta &= 0, & \text{on } \partial\omega^* \end{cases}$$

where  $\omega^* = \bigcup_{\tau} \omega(\tau)$  and by assumptions we have that  $\omega^* \subsetneq \mathbb{S}^{n-1}$ .

**Proposition 4.1.** *Assume that  $\delta, \rho \in (-n - \gamma^* + 2, 0]$ , and*

$$\sup_{\tau \in \mathbb{R}} \{|\sigma(\tau)| + |\sigma'(\tau)| + |\sigma''(\tau)|\} < \varepsilon, \quad (4.1)$$

where  $\varepsilon > 0$  is small enough. Then, for all  $\tau_1 < \tau_2 \in \mathbb{R}$ , and  $R > 0$ , there exists a unique operator

$$G_{\delta, \rho, R, \tau_1, \tau_2} : C_{\delta, \rho}(\Omega_{\tau_1, \tau_2}^R) \rightarrow C_{\delta, \rho}(\Omega_{\tau_1, \tau_2}^R),$$

such that, for each  $f \in C_{\delta,\rho}(\Omega_{\tau_1,\tau_2}^R)$ , the function  $G_{\delta,\rho,R,\tau_1,\tau_2}(f)$  is a solution of problem

$$\begin{cases} \Delta u &= \frac{1}{r_{\sigma(\tau)}^2} f, & \text{in } \Omega_{\tau_1,\tau_2}^R, \\ u &= 0, & \text{on } \partial\Omega_{\tau_1,\tau_2}^R \setminus S_{\tau_1,\tau_2}. \end{cases} \quad (4.2)$$

Moreover the norm of  $G_{\delta,\rho,R,\tau_1,\tau_2}$  is bounded by a constant  $c > 0$  which does not depend on  $R$ ,  $\tau_1$  and  $\tau_2$ .

*Proof.* Without loss of generality we can assume that  $R > 4$ .

We first solve, for each  $r \in (0, \frac{1}{4})$ , the problem

$$\begin{cases} \Delta u &= \frac{1}{|x-\sigma(\tau)|^2} f, & \text{in } \Omega_{\tau_1,\tau_2}^R \setminus \Omega_{\tau_1,\tau_2}^r, \\ u &= 0, & \text{on } \partial(\Omega_{\tau_1,\tau_2}^R \setminus \Omega_{\tau_1,\tau_2}^r). \end{cases} \quad (4.3)$$

and call  $u_r$  its unique solution.

A straightforward calculations show that

$$-\Delta(r_{\sigma(\tau)}^\delta \phi_\delta^*) \geq r_{\sigma(\tau)}^{\delta-2} (1 - |\delta|(|\delta| + 1)|\sigma'|) - |\delta||\sigma''| r_{\sigma(\tau)}^{\delta-1}.$$

We choose  $\varepsilon$  small enough such that

$$-\Delta(r_{\sigma(\tau)}^\delta \phi_\delta^*) \geq \frac{1}{2} (r_{\sigma(\tau)}^{\delta-2} - r_{\sigma(\tau)}^{\delta-1}).$$

Let  $\psi$  be the solution of

$$\begin{cases} \Delta_{\mathbb{S}^{n-1}} \psi = -C\|f\|_{C_{\delta,\rho}(\Omega_{\tau_1,\tau_2}^R)} & \text{in } \omega^* \\ \psi = 0, & \text{on } \partial\omega^* \end{cases}$$

for some constant  $C > 0$  and we define the following cut-of function  $\eta : \mathbb{R}^n \rightarrow [0, 1]$  by  $\eta = 1$  in  $B_{\frac{1}{2}}(0) \subset \mathbb{R}^n$  and  $\eta \in C_0^\infty(B_1(0))$ .

We next set

$$\Phi(\tau, x) = C\|f\|_{C_{\delta,\rho}(\Omega_{\tau_1,\tau_2}^R)} \eta(x) r_{\sigma(\tau)}^\delta \phi_\delta^* + \psi.$$

If we choose the uniform constant  $C > 0$ , large enough, we have by the maximum principle

$$\begin{aligned} |u_r(\tau, x)| &\leq \Phi(\tau, x) \leq C\|f\|_{C_{\delta,\rho}(\Omega_{\tau_1,\tau_2}^R)} \phi_\delta^* |x|^\delta + \psi \\ &\leq C\|f\|_{C_{\delta,\rho}(\Omega_{\tau_1,\tau_2}^R)} \phi_\delta^*(\theta) (|x|^\delta + 1), \quad \forall (\tau, x) \in \Omega_{\tau_1,\tau_2}^R \setminus \Omega_{\tau_1,\tau_2}^r \end{aligned} \quad (4.4)$$

where in the last inequality we have used the fact that

$$\psi(\theta) \leq C\|f\|_{C_{\delta,\rho}(\Omega_{\tau_1,\tau_2}^R)} \phi_\delta^*(\theta), \quad \forall \theta \in \omega^*.$$

Using (4.4) and again the maximum principle we get

$$|u_r(\tau, x)| \leq C\|f\|_{C_{\delta,\rho}(\Omega_{\tau_1,\tau_2}^R)} \phi_\delta^*(\theta) |x|^\delta, \quad \forall (\tau, x) \in \Omega_{\tau_1,\tau_2}^{\frac{1}{2}} \setminus \Omega_{\tau_1,\tau_2}^r. \quad (4.5)$$

Set now  $\psi_0 = \tilde{r}^\rho \phi_\rho^*$ , then

$$\Delta_{\mathbb{S}^{n-1}} \psi_0 = -\tilde{r}^{\rho-2}.$$

Thus using (4.5) and the maximum principle we obtain,

$$|u_r| \leq C(\sup_{\tau \in \mathbb{R}} |\sigma|) \|f\|_{C_{\delta,\rho}(\Omega_{\tau_1,\tau_2}^R)} \|\phi_\rho^*\|_{L^\infty(\omega)} |x|^\rho, \quad \forall r_{\sigma(\tau)} > \frac{1}{2}. \quad (4.6)$$

By standard interior elliptic estimates and Arzela Ascoli theorem, there exists a subsequence  $\{u_{r_j}\}$ , such that  $r_j \downarrow 0$  and  $u_{r_j} \rightarrow u$  locally uniformly. By standard elliptic theory, (4.5) and (4.6), we have that  $u \in C^2(\Omega_{\tau_1,\tau_2}^R)$  and is unique.  $\square$

*Proof of theorem 1.2.* We choose  $\delta = -\frac{2}{p-1}$  and we set

$$u_\varepsilon(x, \tau) = \eta(x) \varepsilon^{-\frac{2}{p-1}} u_1\left(\frac{x - \sigma}{\varepsilon}\right),$$

where  $u_1$  is the function given in theorem 1.1 and  $\eta : \mathbb{R}^n \rightarrow [0, 1]$  is a cut-off function such that  $\eta = 1$  in  $B_{\frac{1}{2}}(0) \subset \mathbb{R}^n$  and  $\eta \in C_0^\infty(B_1(0))$ .

By construction of  $u_1(x)$  and lemma 3.6 we have

$$\begin{aligned} |\nabla_x u_1(\tau, x)| &\leq C(n, p, \lambda, C_{\omega(\tau)}) |x|^{-1} \\ |D_x^2 u_1(\tau, x)| &\leq C(n, p, \lambda, C_{\omega(\tau)}) |x|^{-2}. \end{aligned} \quad (4.7)$$

First we assume that

$$\sup_{\tau \in \mathbb{R}} \{|\sigma(\tau)| + |\sigma'(\tau)| + |\sigma''(\tau)|\} < \tilde{\varepsilon}, \quad (4.8)$$

where  $\tilde{\varepsilon} > 0$  is small enough. Then by the above two estimates (4.7), (4.8) and lemma 3.8 we have

$$|\partial_\tau^2 u_\varepsilon(x, \tau)| \leq C r^{-\frac{2}{p-1}}(\tau) + C(n, \gamma^*) \tilde{\varepsilon} \left( r_{\sigma(\tau)}^{-\frac{2}{p-1}-2} + r_{\sigma(\tau)}^{-\frac{2}{p-2}-1} \right). \quad (4.9)$$

Now, let  $R > 4$ ,  $\tau_1 < \tau_2 \in \mathbb{R}$  and define the following problem

$$\begin{cases} -\Delta u &= u^p, & \text{in } \Omega_{\tau_1, \tau_2}^R, \\ u &> 0, & \text{in } \Omega_{\tau_1, \tau_2}^R \\ u &= 0, & \text{on } \partial\Omega_{\tau_1, \tau_2}^R \setminus S_{\tau_1, \tau_2}. \end{cases} \quad (4.10)$$

We then look for a solution of the form  $u = u_\varepsilon + v$ . By virtue of proposition 4.1 we can rewrite this equation as the fixed point problem

$$v = -G_{\delta, \rho, R, \tau_1, \tau_2} (|x|^2 (\Delta u_\varepsilon + |u_\varepsilon + v|^p)) \quad (4.11)$$

$$\Delta v = -|u_\varepsilon + v|^p - \Delta u_\varepsilon.$$

We assume that  $\varepsilon$  is small enough, then by (4.9) we have for some constant  $C_0(n, \gamma) > 0$ ,

$$\begin{aligned} \| |u_\varepsilon|^p + \Delta u_\varepsilon \|_{C_{\delta, \rho}(\Omega_{\tau_1, \tau_2}^R)} &\leq C_0 \left( \varepsilon^{n+\gamma-2-\frac{p-3}{p-1}} + \varepsilon^2 + \varepsilon + \tilde{\varepsilon} \right) \\ &\leq C_0 (\varepsilon + \tilde{\varepsilon}), \end{aligned}$$

we recall here that  $\delta = -\frac{2}{p-1}$ .

Then, using theorem 1.1 one can easily see that

$$\begin{aligned} &\| |x|^2 |v_\varepsilon + v_1|^p - |v_\varepsilon + v_2|^p \|_{C_{\delta, \rho}(\Omega_{\tau_1, \tau_2}^R)} \\ &\leq C_1(n, \gamma^*, p) \left( \sup_{\tau \in \mathbb{R}} \|\phi_p\|_{L^\infty(\omega)} + \tilde{\varepsilon} \right)^{p-1} \|v_1 - v_2\|_{C_{\delta, \rho}(\Omega_{\tau_1, \tau_2}^1)} \\ &+ C(n, \gamma^*, p) (\varepsilon + \tilde{\varepsilon})^{p-1} \|v_1 - v_2\|_{C_{\delta, \rho}(\Omega_{\tau_1, \tau_2}^R \setminus \Omega_{\tau_1, \tau_2}^1)}, \end{aligned} \quad (4.12)$$

for all  $v_1, v_2 \in C_{\delta, \beta} (C_\omega^R \setminus \{0\} \times (\tau_1, \tau_2))$  such that

$$\|v_i\|_{C_{\delta, \beta}(C_\omega^R \setminus \{0\} \times (\tau_1, \tau_2))} \leq 2C_0(\varepsilon + \tilde{\varepsilon}).$$

We recall that all the constants above do not depend on  $R$ ,  $t_1$ ,  $t_2$ ,  $\varepsilon$  and  $\tilde{\varepsilon}$ . To obtain a contraction mapping is enough to take  $\varepsilon$ ,  $\tilde{\varepsilon}$  small enough and  $p$  close enough to  $\sup_{\tau \in \mathbb{R}} p^*$  to ensure that

$\sup_{\tau \in \mathbb{R}} \|\phi_p(\tau, \cdot)\|_{L^\infty(\omega(\tau))}$  is as small as we need. The above estimates allow an application of contraction mapping principle in the ball of radius  $2C_0(\varepsilon + \tilde{\varepsilon})$  in  $\Omega_{\tau_1, \tau_2}^R$  to obtain a solution to the problem (4.11), which we denote by

$$u_{R, \tau_1, \tau_2} = u_\varepsilon + v_{R, \tau_1, \tau_2}.$$

In view of the fix point argument, we have that  $|v_{R, t_1, t_2}| \leq \frac{u_\varepsilon}{4}$  near  $S_{\tau_1, \tau_2}$ , thus the solution  $u_{R, t_1, t_2}$  is singular along  $S_{\tau_1, \tau_2}$  and positive near  $S_{\tau_1, \tau_2}$ . The maximum principle then implies that

$$u_{R, t_1, t_2} > 0 \quad \text{in} \quad \Omega_{\tau_1, \tau_2}^R.$$

Moreover we have that

$$\|v_{R, \tau_1, \tau_2}\|_{C_{\delta, \beta}(\Omega_{\tau_1, \tau_2}^R)} \leq 2C_0(\varepsilon + \tilde{\varepsilon}).$$

That is,  $v_{R, \tau_1, \tau_2}$  is uniformly bounded by a constant which depend only on  $n, \gamma^*, p$ . By standard interior elliptic estimates and Arzela-Ascoli theorem, there exists a subsequence  $\{u_{R_j, -\tau_j, \tau_j}\}$ , such that  $R_j \uparrow \infty, \tau_j \uparrow \infty$  and  $u_{R_j, -\tau_j, \tau_j} \rightarrow u$  locally uniformly. Again standard elliptic theory yields  $u \in C^2(\Omega_{-\infty, \infty})$ .

For the general case

$$\sup_{\tau \in \mathbb{R}} \{|\sigma(\tau)| + |\sigma'(\tau)| + |\sigma''(\tau)|\} < C,$$

set  $\tilde{\sigma} = \frac{\sigma}{k}$ , where  $k > 0$  is large enough such that

$$\sup_{\tau \in \mathbb{R}} \{|\tilde{\sigma}(\tau)| + |\tilde{\sigma}'(\tau)| + |\tilde{\sigma}''(\tau)|\} < \tilde{\varepsilon}.$$

As before we can find a solution  $u(x)$  of the problem with singularity along  $\{(\tau, x) \in \mathbb{R} \times \mathbb{R}^n : |x - \tilde{\sigma}(\tau)| = 0\}$ . But the function  $v(y) = k^{\frac{2}{p-1}}u(ky)$ , where  $y = kx$ , is a singular solution of the problem and has singularity along  $S_{-\infty, \infty}$ , and the result follows.  $\square$

Let  $\alpha > 0, \Omega$  be a bounded Lipschitz domain such that

$$\Omega \cap \Omega_{\tau_1 - \alpha, \tau_2 + \alpha}^R = \Omega_{\tau_1 - \alpha, \tau_2 + \alpha}^R \subset \mathbb{R}^{n+1}.$$

Let  $C_\delta(\Omega_{\tau_1, \tau_2}^R)$  be the set of continuous function  $f \in C(\Omega_{\tau_1, \tau_2}^R)$  with norm

$$\|f\|_{C_\delta(\Omega_{\tau_1, \tau_2}^R)} = \sup_{(\tau, x) \in \Omega_{\tau_1, \tau_2}^R} \left( r^{-\delta}(\tau) |f| \right).$$

We define  $C_\delta(\Omega)$  to be the space of the continuous function in  $\Omega$  with the norm

$$\|f\|_{C_\delta(\Omega)} = \|f\|_{C_\delta(\Omega_{\tau_1 - \alpha, \tau_2 + \alpha}^R)} + \|f\|_{L^\infty(\overline{\Omega} \setminus \Omega_{\tau_1 - \frac{\alpha}{4}, \tau_2 + \frac{\alpha}{4}}^{\frac{R}{2}})}.$$

We consider a smooth, positive bounded function  $\nu : \overline{\Omega} \rightarrow (0, \infty)$ , which is equal to  $r_{\sigma(\tau)}$  in  $\Omega_{\tau_1 - \frac{\alpha}{4}, \tau_2 + \frac{\alpha}{4}}^{\frac{R}{2}}$  and satisfying

$$0 < \sup_{x \in \overline{\Omega} \setminus \Omega_{\tau_1 - \frac{\alpha}{2}, \tau_2 + \frac{\alpha}{2}}^R} \nu < C.$$

We obtain the following proposition

**Proposition 4.2.** *Let  $\tau_1 < \tau_2 \in \mathbb{R}$  and  $\alpha > 0$  be small enough. Assume that  $\Omega$  is a bounded Lipschitz domain such that*

$$\Omega \cap \Omega_{\tau_1 - 2\alpha, \tau_2 + 2\alpha}^R = \Omega_{\tau_1 - 2\alpha, \tau_2 + 2\alpha}^R \subset \mathbb{R}^{n+1},$$

$\delta \in (-n - \gamma^* + 2, 0]$  and

$$\sup_{\tau \in \mathbb{R}} \{ |\sigma(\tau)| + |\sigma'(\tau)| + |\sigma''(\tau)| \} < \varepsilon, \quad (4.13)$$

for some  $\varepsilon > 0$  small enough. Then, there exists a unique operator

$$G_{\delta, \tau_1, \tau_2} : C_\delta(\Omega) \rightarrow C_\delta(\Omega),$$

such that, for each  $f \in C_\delta(\Omega)$ , the function  $G_{\delta, \tau_1, \tau_2}(f)$  is a solution of the problem

$$\begin{cases} \Delta u &= \frac{1}{\nu^2} f, & \text{in } \Omega, \\ u &= 0, & \text{on } \partial\Omega \setminus S_{\tau_1 - \alpha, \tau_2 + \alpha}. \end{cases} \quad (4.14)$$

Moreover the norm of  $G_{\delta, \tau_1, \tau_2}$  is bounded by a constant  $c > 0$  which does not depend on  $R$ ,  $\tau_1$  and  $\tau_2$ .

*Proof.* Let  $\hat{\sigma}(t)$  be a bounded smooth curve such that

$$\sup_{\tau \in \mathbb{R}} \{ |\hat{\sigma}(\tau)| + |\hat{\sigma}'(\tau)| + |\hat{\sigma}''(\tau)| \} < 2\varepsilon,$$

$$\begin{aligned} r_{\hat{\sigma}(\tau)} &= r_{\sigma(\tau)}, & \forall (\tau, x) \in \Omega_{\tau_1 - \frac{\alpha}{4}, \tau_2 + \frac{\alpha}{4}}^R, \\ r_{\hat{\sigma}(\tau)} &\geq r_{\sigma(\tau)}, & \forall (\tau, x) \in \Omega, \end{aligned}$$

and

$$r_{\hat{\sigma}(\tau)} > c > 0, \quad \forall (\tau, x) \in \Omega_{\tau_1 - \alpha, \tau_2 + \alpha}^R \setminus \overline{\Omega_{\tau_1 - \frac{\alpha}{2}, \tau_2 + \frac{\alpha}{2}}^R}.$$

Given  $\tau$ , we let  $\hat{\omega}(\tau) \subsetneq \mathbb{S}^{n-1}$  be the corresponding Lipschitz spherical cap and  $(r_{\hat{\sigma}(\tau)}, \theta) \in [0, \infty) \times \mathbb{S}^{n-1}$  be the spherical-coordinates of  $x \in \mathbb{R}^n$  centered at  $\hat{\sigma}(\tau)$  abbreviated by  $x = (r_{\hat{\sigma}(\tau)}, \theta)$ .

We set

$$\begin{aligned} \hat{C}_{\hat{\omega}(\tau)} &= \{ (r_{\hat{\sigma}(\tau)}, \theta) : \hat{r}(\tau) > 0, \theta \in \hat{\omega}(\tau) \}, \\ \hat{\Omega}_{\tau_1, \tau_2} &= \{ (\tau, x) \in (\tau_1, \tau_2) \times \mathbb{R}^n : x \in \hat{C}_{\hat{\omega}(\tau)} \} \end{aligned}$$

and  $\hat{\Omega}_{\tau_1, \tau_2}^R = \hat{\Omega}_{\tau_1, \tau_2} \cap \{ (\tau, x) \in (\tau_1, \tau_2) \times \mathbb{R}^n : x \in B_R(\hat{\sigma}(\tau)) \} \subset \mathbb{R}^{n+1}$ . We construct  $\hat{\omega}(\tau)$  such that

$$\Omega_{\tau_1 - \alpha, \tau_2 + \alpha}^R \subsetneq \hat{\Omega}_{\tau_1 - \alpha, \tau_2 + \alpha}^{2R},$$

$$\hat{\Omega}_{\tau_1 - \frac{\alpha}{4}, \tau_2 + \frac{\alpha}{4}}^R = \Omega_{\tau_1 - \frac{\alpha}{4}, \tau_2 + \frac{\alpha}{4}}^R.$$

We next define  $\eta$  be a cut-off function satisfying  $\eta = 1$  in  $\Omega_{\tau_1 - \frac{\alpha}{2}, \tau_2 + \frac{\alpha}{2}}^{\frac{R}{2}}$  and  $\eta = 0$  in  $\Omega \setminus \Omega_{\tau_1 - \alpha, \tau_2 + \alpha}^R$ .

We write  $\hat{f} = \eta f$  and we let  $u_1 = G_{\delta, \rho, R, \tau_1, \tau_2}(\hat{f})$  be the function given by proposition 4.1 in  $\hat{\Omega}_{\tau_1 - \alpha, \tau_2 + \alpha}^{2R}$ .

Set

$$\tilde{f} = f - \nu \Delta(\eta u_1),$$

then  $\tilde{f}$  has support in  $\Omega \setminus \Omega_{\tau_1 - \frac{\alpha}{4}, \tau_2 + \frac{\alpha}{4}}^{\frac{R}{2}}$ , and  $\tilde{f} \in C(\Omega)$ . Furthermore we have

$$\|\tilde{f}\|_{C_\delta(\Omega)} \leq C \|f\|_{C_\delta(\Omega)},$$

for some positive constant  $C > 0$ .

Finally, let  $u_2$  be a solution of

$$\begin{cases} \Delta u &= \frac{1}{\nu^2} \tilde{f}, & \text{in } \Omega, \\ u &= 0, & \text{on } \partial\Omega, \end{cases}$$

which clearly satisfy the bound

$$\|u_2\|_{L^\infty(\Omega)} \leq C \|\tilde{f}\|_{C_\delta(\Omega)} \leq C \|f\|_{C_\delta(\Omega)}.$$

The desired result then follows by looking for a solution of (4.14) of the form  $u = \eta u_1 + u_2$ .  $\square$

*Proof of theorem 1.3.* We choose  $\delta = -\frac{2}{p-1}$  and we set

$$u_\varepsilon(x, \tau) = \eta(x) \varepsilon^{-\frac{2}{p-1}} u_1\left(\frac{x - \sigma}{\varepsilon}\right),$$

where  $u_1$  is the function given by theorem 1.1 and  $\eta : \mathbb{R}^n \rightarrow [0, 1]$  is a cut-of function such that  $\eta = 1$  in  $\Omega_{\tau_1 - \frac{\alpha}{2}, \tau_2 + \frac{\alpha}{2}}^{\frac{R}{2}}$  and  $\eta = 0$  in  $\Omega \setminus \Omega_{\tau_1 - \alpha, \tau_2 + \alpha}^R$ .

The rest of the proof is the same as in theorem 1.2, the only difference is that we use proposition 4.2 instead of proposition 4.1.  $\square$

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Proof of lemma 2.1 To prove lemma 2.1 we need the following inequality whose the proof can be found in [10] (theorem 2, page 43).

**Lemma .3.** *Let  $A(r)$ ,  $B(r)$  be nonnegative functions such that  $1/A(r)$ ,  $B(r)$  are integrable in  $(r, \infty)$  and  $(0, r)$ , respectively, for all positive  $r < \infty$ . Then, for  $q \geq 2$  the Sobolev inequality*

$$\left[ \int_0^s B(t) |u(t)|^q dt \right]^{1/q} \leq C \left[ \int_0^s A(t) |u'(t)|^2 dt \right]^{1/2}, \quad (15)$$

is valid for all  $u \in C^1[0, s]$  such that  $u(s) = 0$  (or vanish near infinity, if  $s = \infty$ ), if and only if

$$K = \sup_{r \in (0, s)} \left[ \int_0^r B(t) dt \right]^{1/q} \left[ \int_r^s (A(t))^{-1} dt \right]^{1/2}$$

is finite. The best constant in (15) satisfies the following inequality

$$K \leq C \leq K \left( \frac{q}{q-1} \right)^{1/2} q^{1/q}.$$

*Proof of lemma 2.1.* Let  $n \geq 3$ , (for  $n = 2$  the proof is easy and we omit it). By our assumptions on  $\omega(\tau)$  and without loss of generality, we can set  $\theta_1 = \cos t$ , with  $0 < t < \beta(\tau)$ , where  $\beta(\tau)$  is a smooth function with bounded derivatives such that

$$0 < \inf_{\tau \in \mathbb{R}} \beta(\tau) < \sup_{\tau \in \mathbb{R}} \beta(\tau) < \pi.$$

Then problem (2.1) is clearly equivalent to

$$\begin{cases} -\sin^{2-n} t \frac{d}{dt} \left( \sin^{n-2} t \frac{d\phi_1}{dt} \right) &= \lambda \phi_1, & \text{in } (0, \beta(\tau)). \\ \phi_1(\beta(\tau)) &= 0 \\ \partial_t \phi_1(0) &= 0. \end{cases} \quad (.16)$$

We denote by  $\mathcal{H}((0, \beta(\tau)))$  the completion of  $C^\infty([0, \beta(\tau)])$  under the norm

$$\|v\|_{\mathcal{H}((0, \beta(\tau)))}^2 = \int_0^{\beta(\tau)} \sin^{n-2}(t) |\partial_t v|^2 dt < \infty,$$

and the property  $v(\beta(\tau)) = \partial_t v(0) = 0$ .

The space  $\mathcal{H}(\omega(\tau))$  is a Hilbert space with inner product

$$(u, v) = \int_0^{\beta(\tau)} \sin^{n-2}(t) \partial_t u \partial_t v dt.$$

Indeed, by lemma .3 and our assumptions on  $\beta(\tau)$ , we can easily obtain that

$$\int_0^{\beta(\tau)} v^2 \sin^{n-3} t dt \leq C(n) \int_0^{\beta(\tau)} \sin^{n-2}(t) |\partial_t v|^2 dt. \quad (.17)$$

By above inequality we can prove that the space  $\mathcal{H}(\omega(\tau))$  is compactly embedded in

$$L_{\sin t}^2((0, \beta(\tau))) := \left\{ u : (0, \beta(\tau)) \rightarrow \mathbb{R} : \int_0^{\beta(\tau)} u^2 \sin^{n-2}(t) dt < \infty \right\}.$$

Thus using standard arguments we can prove that the eigenvalue problem

$$0 < \lambda(\tau) = \inf_{u \in \mathcal{H}((0, \beta(\tau)))} \frac{\int_0^{\beta(\tau)} \sin^{n-2}(t) \left| \frac{du}{dt} \right|^2 dt}{\int_0^{\beta(\tau)} u^2 \sin^{n-2}(t) dt},$$

has a positive minimizer  $\phi_1(\tau, t) \in \mathcal{H}(0, \beta(\tau))$ .

But,

$$\begin{aligned} C(n) \int_0^{\beta(\tau)} \sin^{n-2}(t) |\partial_t \phi_1|^2 dt &= \int_\omega |\nabla \phi_1|^2 dS, \\ C(n) \int_0^{\beta(\tau)} \sin^{n-2}(t) |u|^2 dt &= \int_\omega |\phi_1|^2 dS = 1, \end{aligned} \quad (.18)$$

thus  $\phi_1 \in H_0^1(\omega(\tau))$  and is a weak solution of the eigenvalue problem (2.1). Hence by standard elliptic arguments we can prove that  $\phi_1 \in L^\infty(\omega(\tau))$ . In addition by our assumption we have that

$$\sup_{\tau \in \mathbb{R}} \sup_{t \in (0, \beta(\tau))} |\phi_1(\tau, t)| < C. \quad (.19)$$

By the ODE equation (.16) and the estimate (.19), we can write

$$\phi_1(\tau, t) = \lambda \int_t^{\beta(\tau)} \frac{1}{\sin^{n-2} s} \int_0^s \sin^{n-2}(r) \phi_1(\tau, r) dr ds. \quad (.20)$$



Thus we have the following estimates

$$\begin{aligned} \sup_{\tau \in \mathbb{R}} \sup_{t \in (0, \beta(\tau))} \left| \frac{1}{\sin t} \partial_t \phi_1(\tau, t) \right| &\leq C \sup_{\tau \in \mathbb{R}} \sup_{t \in (0, \beta(\tau))} |\phi_1(\tau, t)| \\ \sup_{\tau \in \mathbb{R}} \sup_{t \in (0, \beta(\tau))} |\partial_t^2 \phi_1(\tau, t)| &\leq C \sup_{\tau \in \mathbb{R}} \sup_{t \in (0, \beta(\tau))} |\phi_1(\tau, t)|. \end{aligned} \quad (.21)$$

Setting now  $s = \frac{t}{\beta(\tau)}$ , we have that  $\phi_1 = \phi_1(\tau, s)$  satisfies

$$\begin{cases} \frac{1}{\beta^2(\tau)} \partial_s^2 \phi_1(\tau, s) + \frac{(n-2) \cos(\beta(\tau)s)}{\beta(\tau) \sin(\beta(\tau)s)} \partial_s \phi_1(\tau, s) + \lambda(\tau) \phi_1(\tau, s) &= 0 & \text{in } (0, 1) \\ \phi_1(1) &= 0 \\ \partial_t \phi_1(0) &= 0. \end{cases}$$

It is easy to see that  $\lim_{h \rightarrow 0} \phi_1(\tau + h, s) = \phi_1(\tau, s)$  in  $L^\infty(\mathbb{R} \times (0, 1))$ . We set

$$u_h(\tau) = \frac{\phi_1(\tau + h, s) - \phi_1(h, s)}{h}, \quad \phi_1(\tau) = \phi_1(\tau, t),$$

then  $u_h$  satisfies

$$\begin{aligned} &\frac{1}{\beta^2(\tau + h)} \partial_s^2 u_h(\tau) + \frac{(n-2) \cos(\beta(\tau + h)s)}{\beta(\tau + h) \sin(\beta(\tau + h)s)} \partial_s u_h(\tau) + \lambda(\tau + h) u_h(\tau) \\ &= - \frac{\frac{1}{\beta^2(\tau + h)} - \frac{1}{\beta^2(\tau)}}{h} \partial_s^2 \phi_1(\tau) - \frac{\lambda(\tau + h) - \lambda(\tau)}{h} \phi_1(\tau) \\ &\quad - (n-2) \frac{\frac{\cos(\beta(\tau + h)s)}{\beta(\tau + h) \sin(\beta(\tau + h)s)} - \frac{\cos(\beta(\tau)s)}{\beta(\tau) \sin(\beta(\tau)s)}}{h} \partial_s \phi_1(\tau) = F_h(\tau, s), \end{aligned} \quad (.22)$$

with  $u_h(\tau, 1) = \partial_s u_h(\tau, 0) = 0$ . On the other hand notice that

$$\begin{aligned} &\sup_{\tau \in \mathbb{R}} \left| (n-2) \frac{\frac{\cos(\beta(\tau + h)s)}{\beta(\tau + h) \sin(\beta(\tau + h)s)} - \frac{\cos(\beta(\tau)s)}{\beta(\tau) \sin(\beta(\tau)s)}}{h} \partial_s \phi_1(\tau, s) \right| \\ &\leq \sup_{\tau \in \mathbb{R}} \left| (n-2) \left( -\frac{\beta'(\tau)}{\beta^2(\tau)} \cot(\beta(\tau)s) - \frac{s\beta'(\tau)}{\sin^2 \beta(\tau)s} \right) \partial_s \phi_1(\tau, s) \right| \\ &< C(n, \inf_{\tau \in \mathbb{R}} \beta(\tau)), \end{aligned} \quad (.23)$$

where in the last inequality we have used (.21) and our assumptions on  $\beta$ . Also using our assumption on  $\lambda$  we have that

$$\sup_{h \in \mathbb{R}} \sup_{\tau \in \mathbb{R}} F_h(\tau, s) < C(n, \inf_{\tau \in \mathbb{R}} \beta(\tau)). \quad (.24)$$

Finally combining above estimates(.22)-(.24) we have

$$\lim_{h \rightarrow 0} \sup_{\tau \in \mathbb{R}} \int_0^1 u_h^2(\tau, s) \sin^{n-2}(\beta(\tau)s) ds < C < \infty. \quad (.25)$$

By (.25) we can prove

$$\sup_{\tau \in \mathbb{R}} \sup_{\tau \in \omega(\tau)} |u_h| < C$$

and we have the following representation formula

$$\begin{aligned} \frac{u_h(\tau, s)}{\beta^2(\tau + h)} &= \lambda(\tau + h) \int_s^1 \frac{1}{\sin^{n-2}(\beta(\tau + h)\xi)} \int_0^\xi \sin^{n-2}(\beta(\tau + h)r) u_h(\tau, r) dr d\xi \\ &\quad - \int_s^1 \frac{1}{\sin^{n-2}(\beta(\tau + h)\xi)} \int_0^\xi \sin^{n-2}(\beta(\tau + h)r) F_h(\tau, r) dr d\xi. \end{aligned}$$

The rest of the proof is standard and we omit it.  $\square$

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